Matrix Manipulations for Earth Scientists

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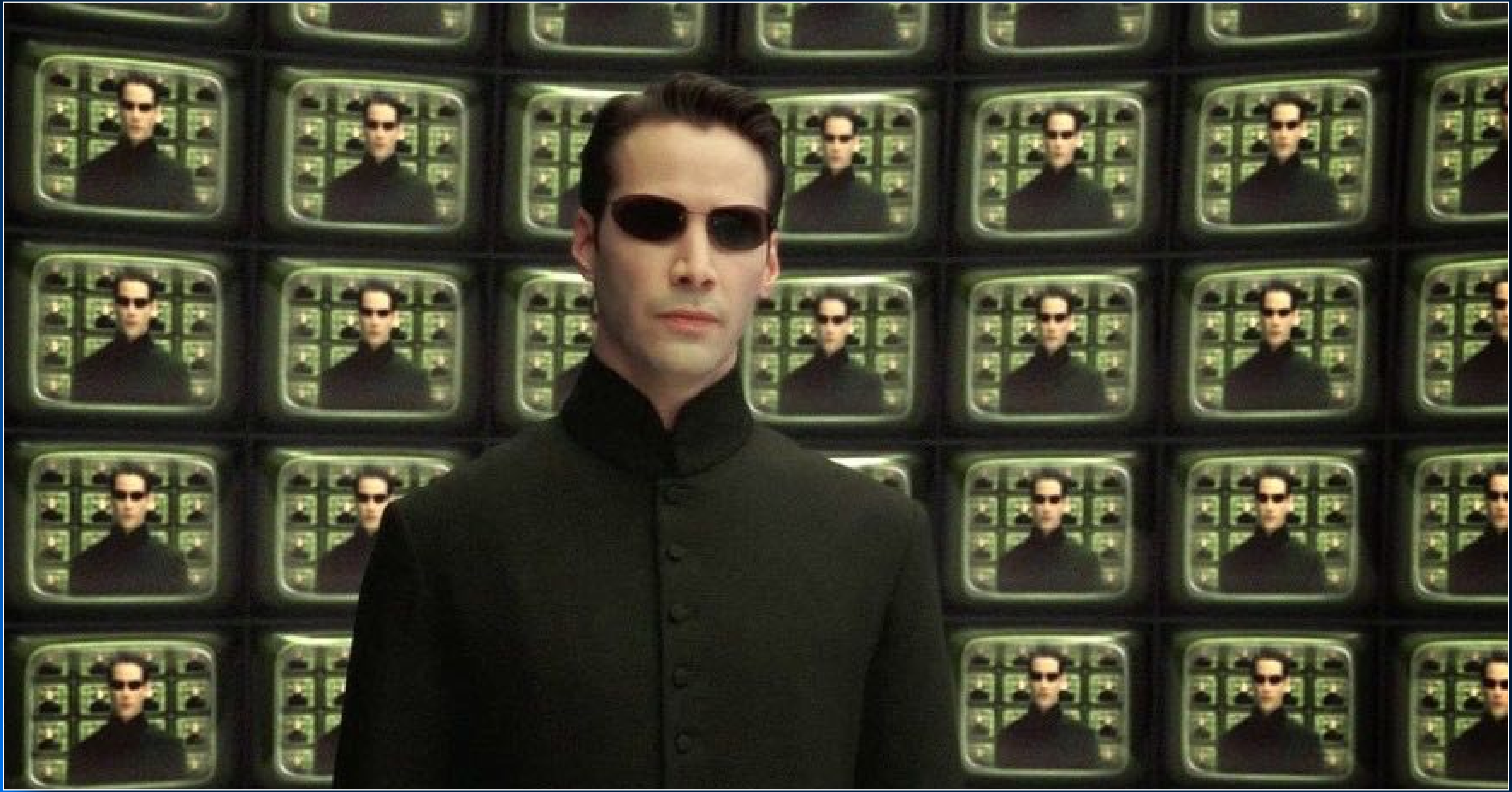
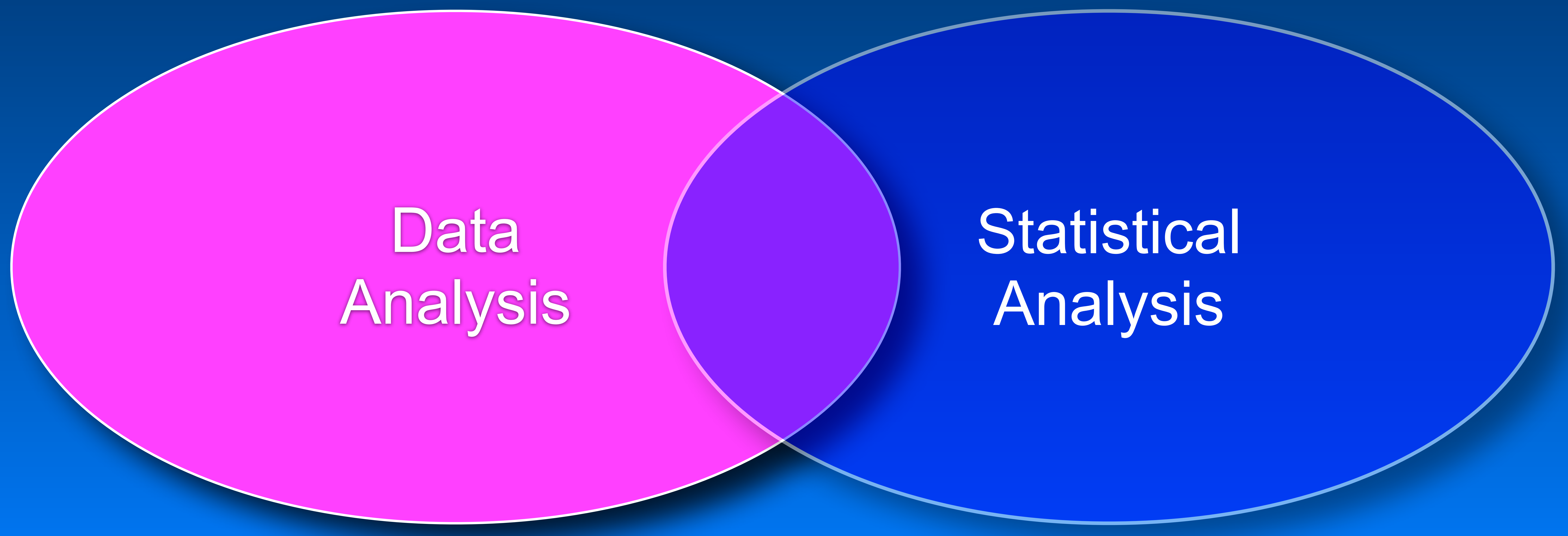


What is Multivariate Data Analysis?

Data Analysis - The act of examining a set of data for informative patterns or trends on which decisions/interpretations can be made.

Statistical Analysis - The comparison of data with predictions of a reasonable null hypothesis/model for the purpose of determining the the level of confidence, likelihood or probability a particular interpretation is correct.



What is Multivariate Data Analysis?

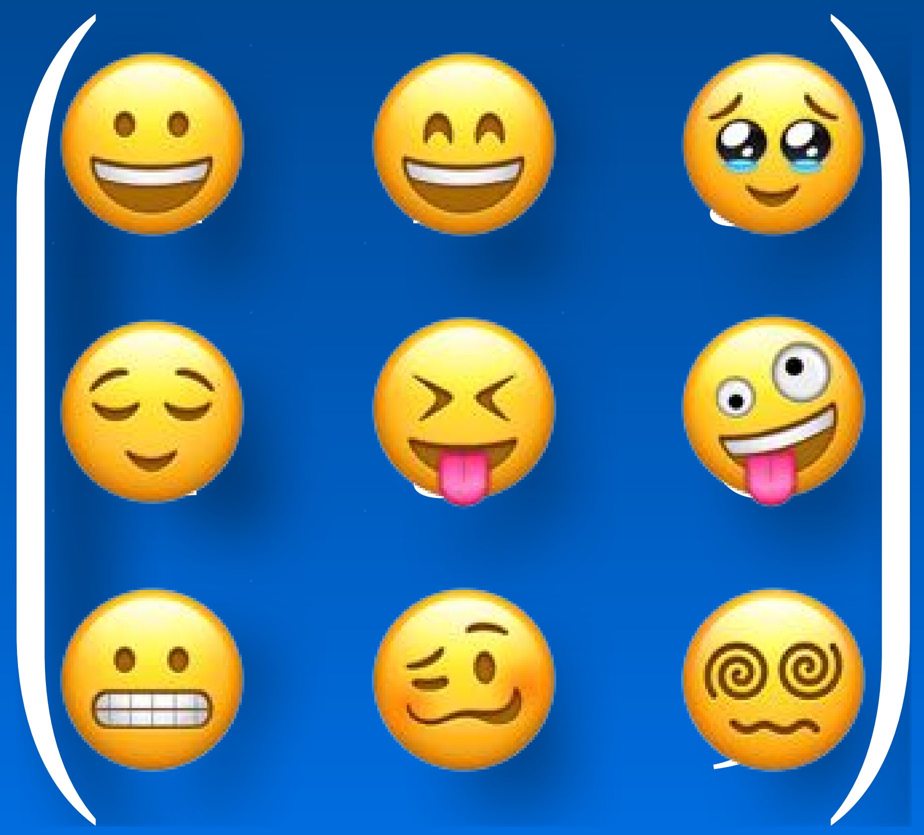
Data Analysis - The act of examining a set of data for informative patterns or trends on which decisions/interpretations can be made.

- Univariate - The analysis of single variables.
  Usually takes the form of a dataset's description, typically using distributions, measures of central tendency, and measures of variance,
- Bivariate - The analysis of paired variables.
  Usually takes the form of trend analysis to reveal how the variable's joint variation is structured, typically using regression analysis
- Multivariate - The simultaneous analysis of three or more variables.
  Can take make different forms (e.g., co-directionality, co-variance, clustering), but is usually performed for the purpose of revealing how the variable's joint variation is structured.

Matrices

Matrix - a rectangular grid or array of numbers, symbols, or expressions arranged in rows and columns. Matrices are fundamental tools in mathematics for data organization and linear algebra. Matrices algebra is employed extensively for efficient calculation, transformation, and representation of systems of equations.

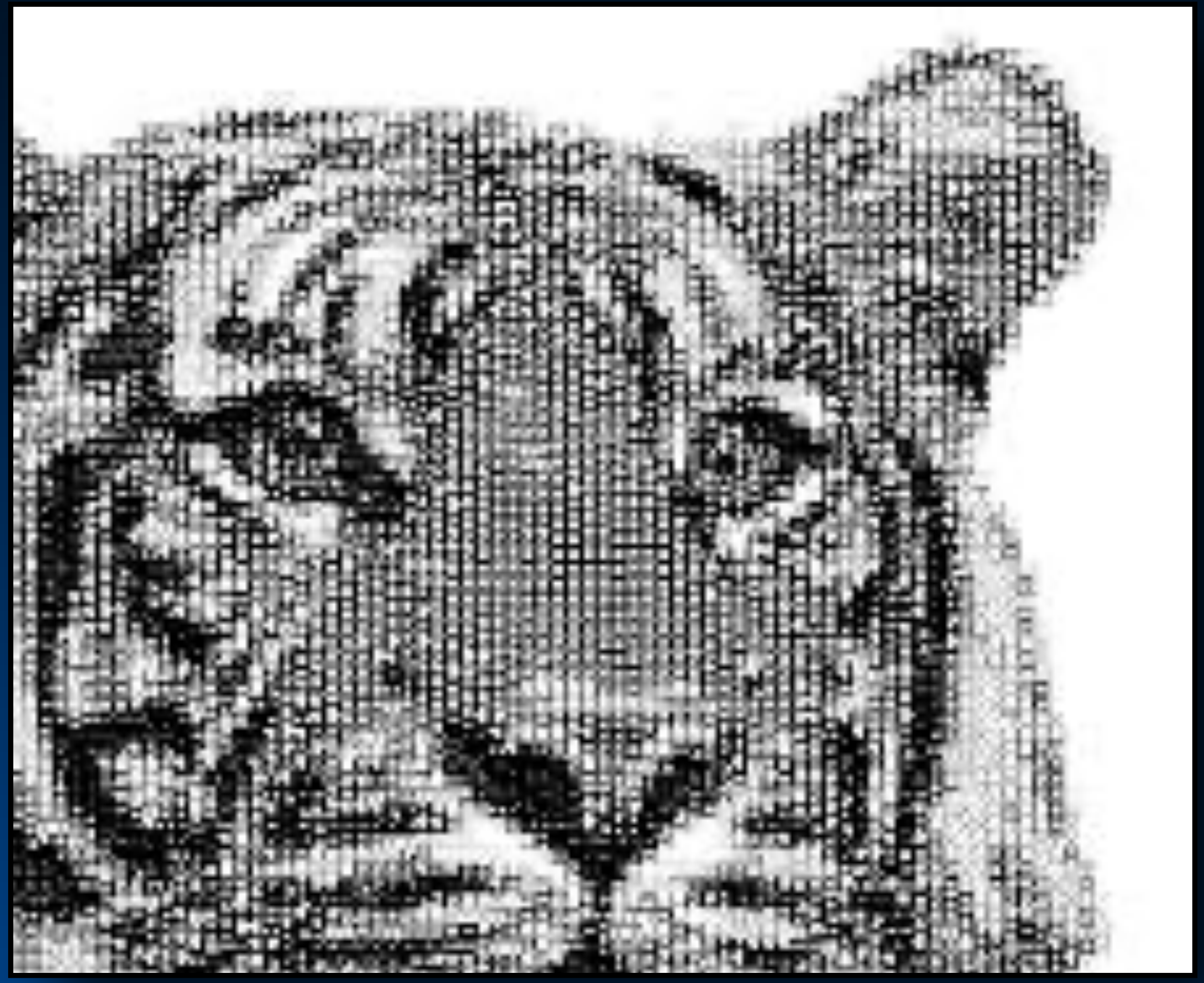
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$



$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \kappa \\ \sigma & \tau & \zeta \end{pmatrix}$$

Matrices

1966



1976

2006

2026

Cartesian System & Matrices

Cartesian Coordinate System

9 x 9

Digital Image

0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	1	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0

9 x 9

Cartesian System & Matrices

Cartesian Coordinate System

9 x 9

Digital Image

0	0	0	0	0	0	0	0
0	0	0			0	0	0
0	0	0			0	0	0
0							0
0							0
0	0					0	0
0	0					0	0
0	0		0	0		0	0
0	0	0	0	0	0	0	0

9 x 9

Cartesian System & Matrices

Cartesian Coordinate System

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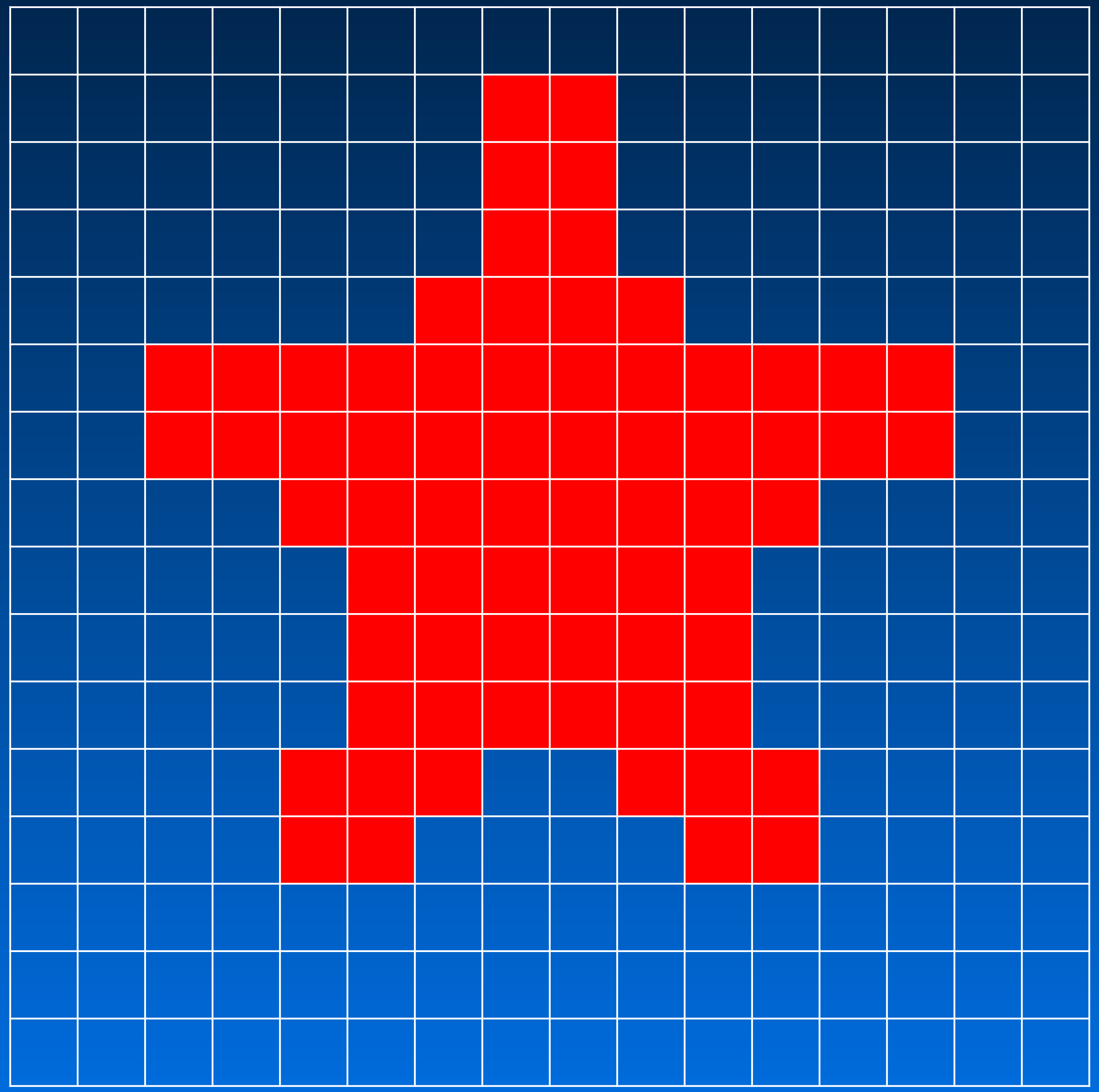
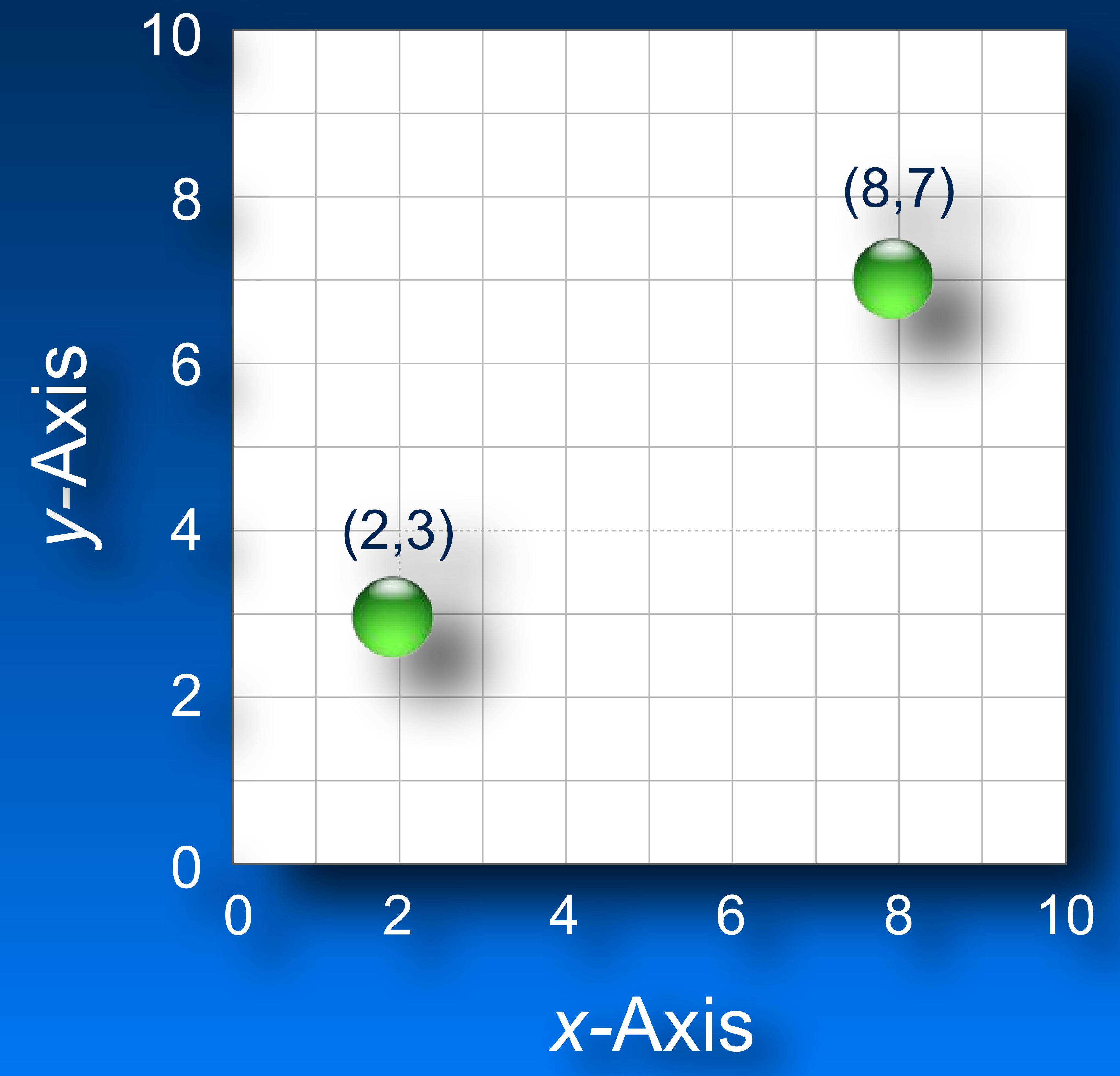
Digital Image



16 x 16

Matrices & Variables

Scalar - a single element of a field or range of variation which may be used to define a variable/vector space.



Matrices & Variables

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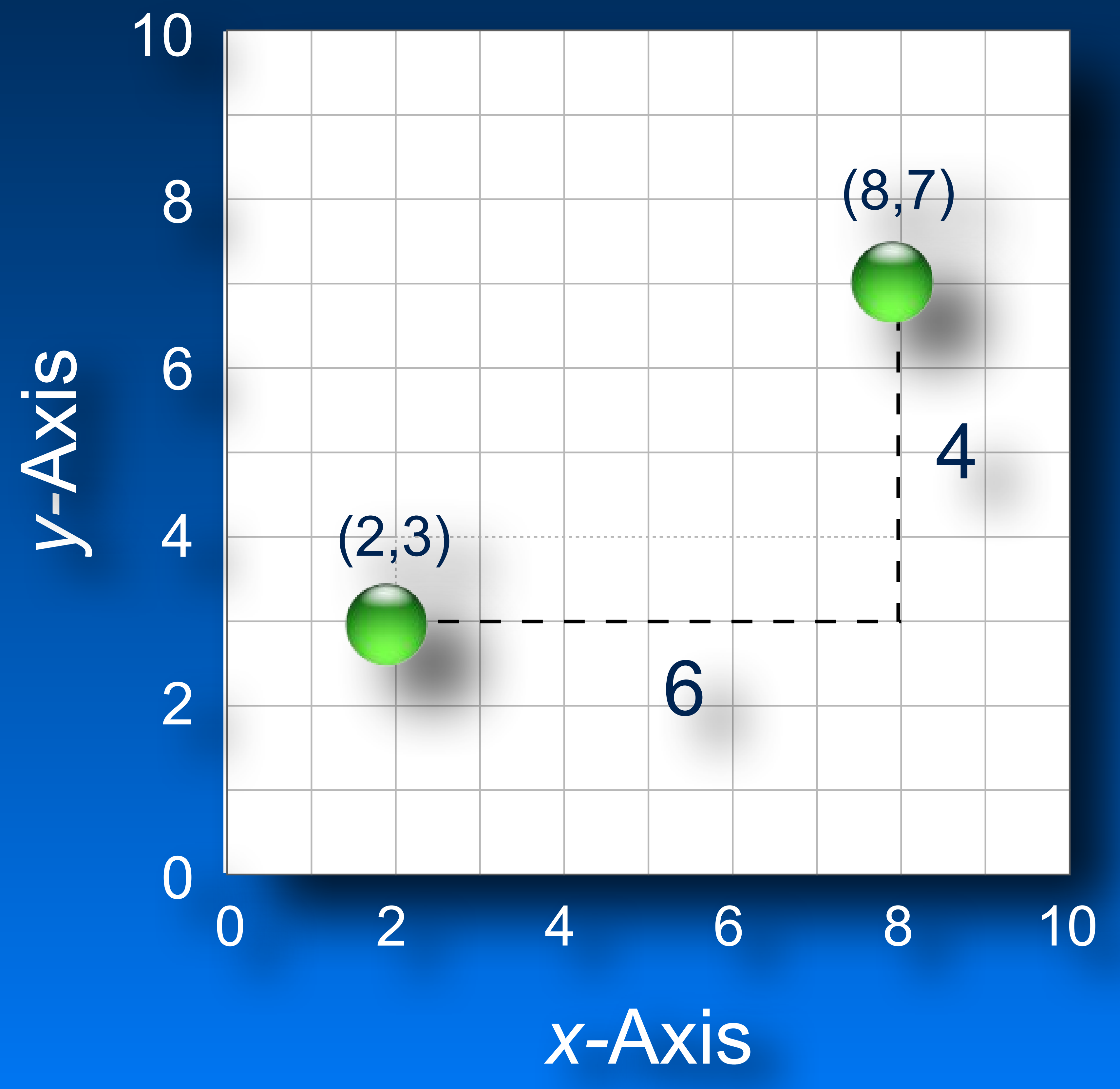
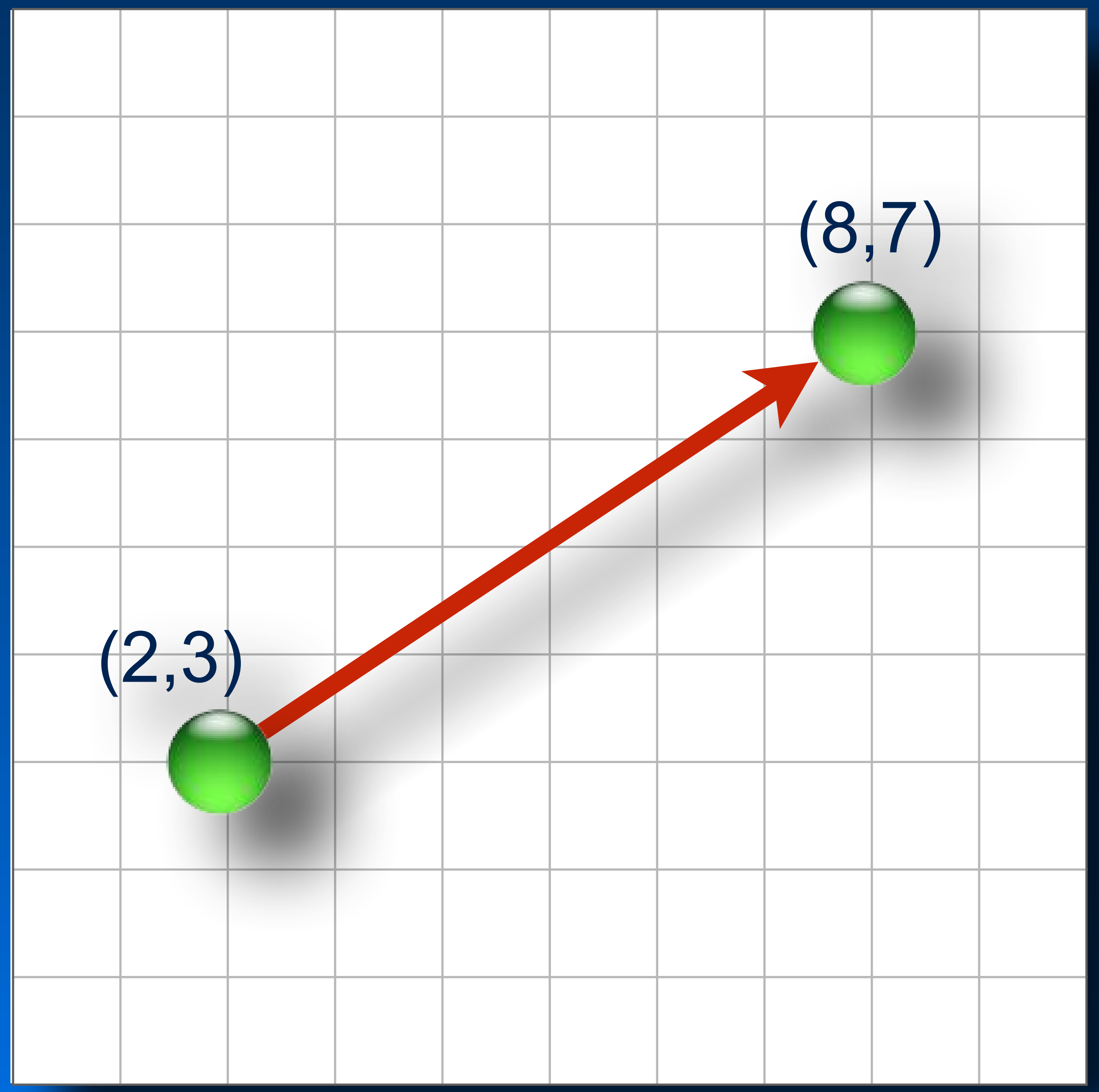


$$\Delta x = x_2 - x_1$$
$$\Delta x = 8 - 2$$
$$\Delta x = 6$$

$$\Delta y = y_2 - y_1$$
$$\Delta y = 7 - 3$$
$$\Delta y = 4$$

Matrices & Variables

Vector - a geometric object that has magnitude and direction, both of which are derived from a set of scalar values.



Matrices & Variables

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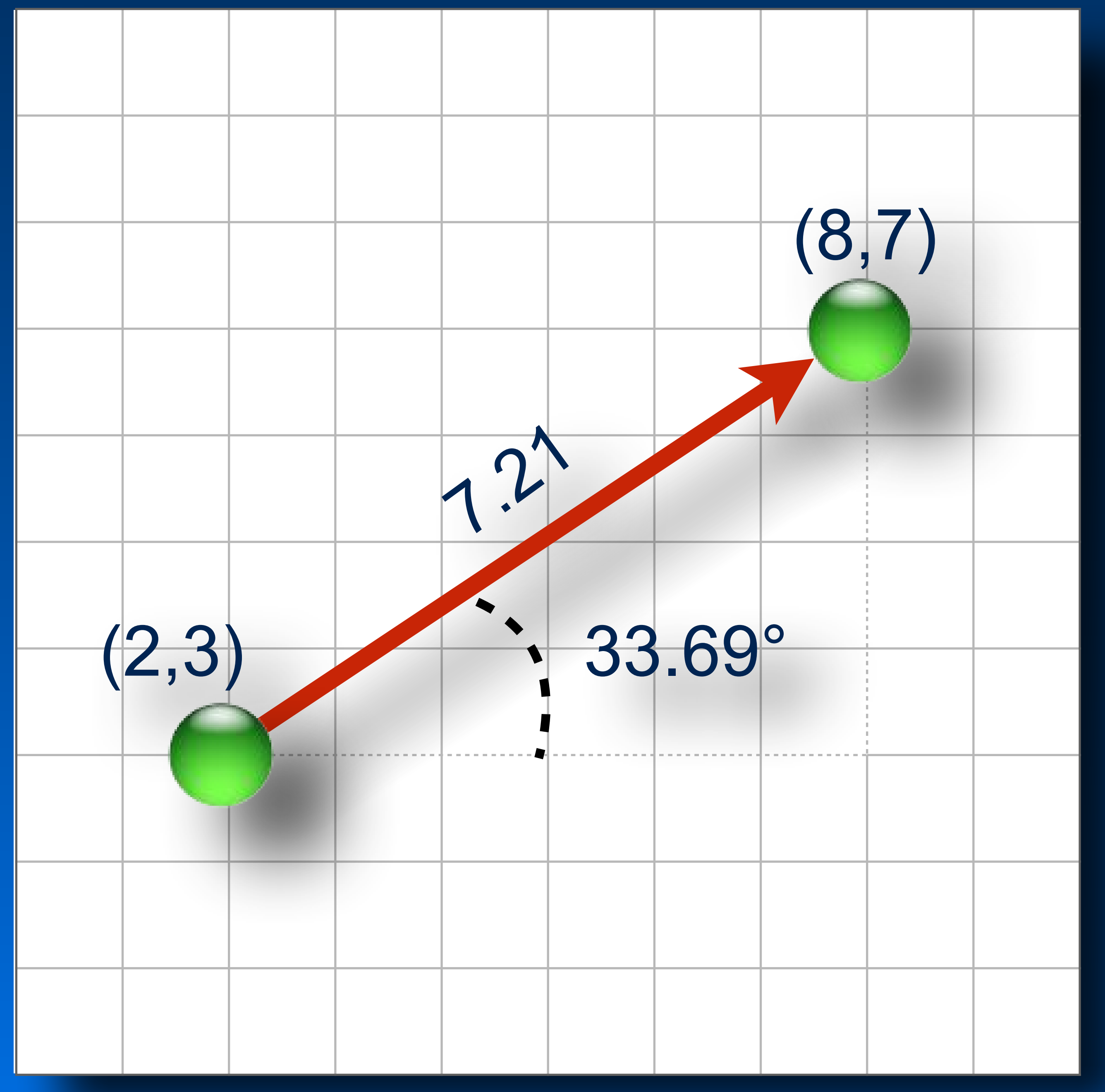


$$d = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$$
$$d = ((8 - 2)^2 + (7 - 3)^2)^{1/2}$$
$$d = ((6)^2 + (4)^2)^{1/2}$$
$$d = 7.2111$$

$$d = \arctan((y_2 - y_1)/(x_2 - x_1))$$
$$d = \arctan((7-3)/(8-2))$$
$$d = \arctan(4/6)$$
$$d = 33.6901°$$

Matrix Nomenclature

Size - Number of matrix cells, usually expressed as the number of columns x the number of rows y.

5	55	94	84	2	17	58
26	93	73	54	33	68	9
93	41	38	18	74	69	62
59	95	100	28	31	98	24
80	12	94	67	69	89	19

5 x 7 Matrix

13	19	38	82	55	6	87	58	66	88
72	20	38	45	59	69	61	38	33	56
23	29	40	15	46	17	51	38	51	44
41	63	67	45	53	65	78	48	23	12
68	91	58	88	5	17	35	84	90	32
54	80	54	52	32	83	23	70	40	38
46	55	23	87	55	92	96	17	31	85
45	84	18	2	22	3	37	2	66	83
87	3	66	13	68	95	7	25	9	30
48	99	68	59	32	13	88	4	36	67

10 x 10 Matrix

Matrix Nomenclature

Shape - Ratio of the number of rows and number of columns. If this ratio = 1.0 the matrix is square. If any other value, the matrix is rectangular.

21	52	87	35	76	39	78
29	72	57	80	78	99	94
38	47	43	28	53	99	44
58	77	36	42	93	30	12
58	52	66	80	70	66	24

Rectangular Matrix

16	6	9	55	42	40	42	56	68	20
91	59	65	91	15	47	22	64	19	35
10	74	74	12	79	80	81	27	75	3
71	3	7	15	40	62	93	13	64	83
56	52	53	19	8	36	44	48	100	98
66	15	16	3	74	58	98	42	78	28
4	36	44	40	51	94	25	79	30	38
10	97	66	46	39	93	20	62	94	28
52	97	39	98	0	76	28	80	65	55
82	38	10	40	42	16	79	64	4	39

Square Matrix

Matrix Nomenclature

Symmetry - a square matrix whose entries are identical across its trace.

Non-Symmetric, Square Random Matrix

98	68	35	43	50	53	3	45	57	42
66	14	32	55	62	91	63	3	89	20
11	45	14	70	15	80	63	7	33	44
48	41	15	52	26	42	45	79	16	41
47	86	72	82	30	61	28	51	85	95
89	96	2	30	99	63	54	2	75	88
100	64	22	10	43	96	67	63	81	77
18	100	43	68	52	70	39	76	43	16
57	90	73	6	61	66	15	24	5	53
3	60	44	75	21	20	62	7	45	79

Symmetric, Square Random Matrix

81	44	71	86	50	61	54	13	2	28
44	83	70	42	93	71	76	39	24	90
71	70	14	39	6	37	91	84	36	11
86	42	39	98	0	7	57	36	61	50
50	93	6	0	86	54	88	11	99	29
61	71	37	7	54	60	9	72	75	100
54	76	91	57	88	9	25	93	95	51
13	39	84	36	11	72	93	2	83	87
2	24	36	61	99	75	95	83	59	7
28	90	11	50	29	100	51	87	7	13

Matrix Nomenclature

Symmetry - a square matrix whose entries are identical across its trace.

Non-Symmetric, Square Random Matrix

93	98	9	8	98	85	82	51	68	44
18	4	76	84	3	2	1	65	42	3
88	94	1	67	68	42	84	16	33	95
48	49	16	94	13	64	93	31	27	41
21	48	93	76	81	28	66	32	31	77
6	54	23	39	49	54	34	39	23	74
31	52	54	15	86	27	38	23	48	91
46	98	94	42	1	6	22	39	16	18
95	29	83	58	29	94	93	87	0	87
49	64	9	51	82	45	96	68	72	62

Symmetric, Square Random Matrix

6	62	8	23	31	52	15	48	80	50
62	71	18	66	88	22	28	15	97	7
8	18	86	51	46	8	77	59	63	85
23	66	51	89	3	75	50	71	2	27
31	88	46	3	67	33	61	37	12	13
52	22	8	75	33	99	51	50	4	90
15	28	77	50	61	51	4	19	95	5
48	15	59	71	37	50	19	74	37	58
80	97	63	2	12	4	95	37	81	90
50	7	85	27	13	90	5	58	90	35

Matrix Nomenclature

Trace & Diagonals - The trace of a square matrix subdivides the matrix into two diagonals (upper & lower).

Trace of a Random Matrix

81	15	0	10	5	43	48	96	90	1
48	8	25	92	53	54	57	67	24	91
64	69	64	97	90	94	45	49	33	14
0	52	69	74	23	8	83	62	20	17
73	4	46	44	40	21	69	24	15	68
90	82	43	76	51	1	9	87	27	7
3	78	81	45	25	66	66	76	63	47
54	42	78	78	100	33	3	7	25	14
4	38	42	20	11	41	48	41	49	11
20	2	36	17	84	26	98	16	67	89

Upper & Lower Triangles Of a Random Matrix

42	33	1	1	76	93	69	82	15	96
17	33	11	90	25	98	30	88	85	18
79	98	91	41	77	93	51	19	54	23
84	4	60	84	47	60	96	36	10	70
53	15	37	37	54	39	69	89	15	80
74	13	20	72	52	40	50	59	69	32
70	93	3	17	59	52	91	96	74	58
44	24	1	83	62	6	7	86	83	10
47	63	57	76	13	54	79	59	84	5
47	45	37	41	23	57	2	0	12	84

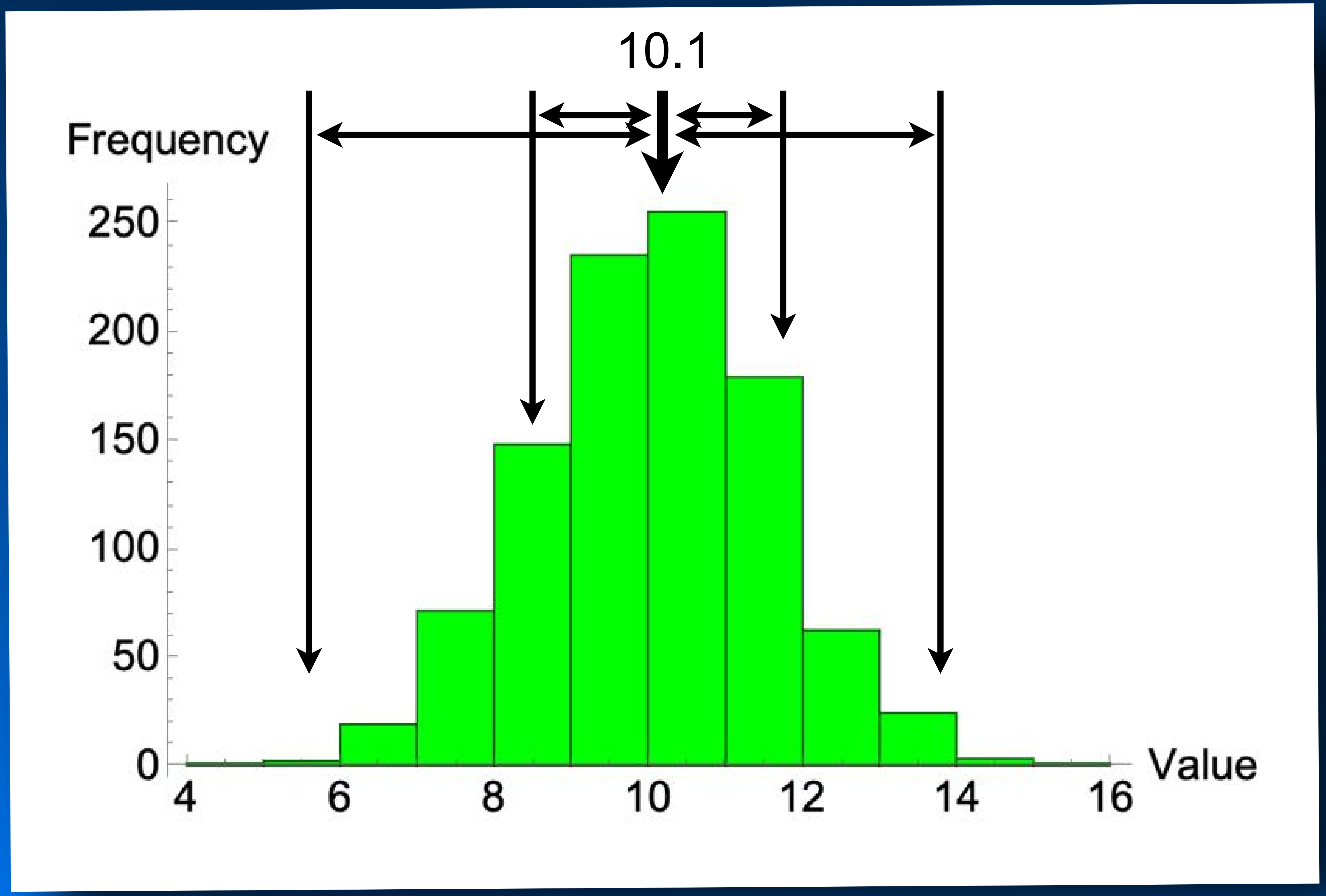
Matrices as Data Storage

Data Matrix - One of the primary uses of matrices is to hold data. In standard format objects or cases or placed in the matrix's rows and variables in its columns. Typically, data matrices are rectangular.

		Variables									
		1	2	3	4	5	6	7	8	9	10
Objects	1	28	73	22	24	36	22	52	0	55	10
	2	11	39	54	10	3	25	58	55	45	70
	3	70	86	72	84	36	7	57	43	70	53
	4	20	94	94	23	41	57	81	61	97	86
	5	15	100	22	22	67	13	67	24	20	80

Statistics as Analysis

Variance - the average of the squared deviations from the arithmetic mean of a variable.

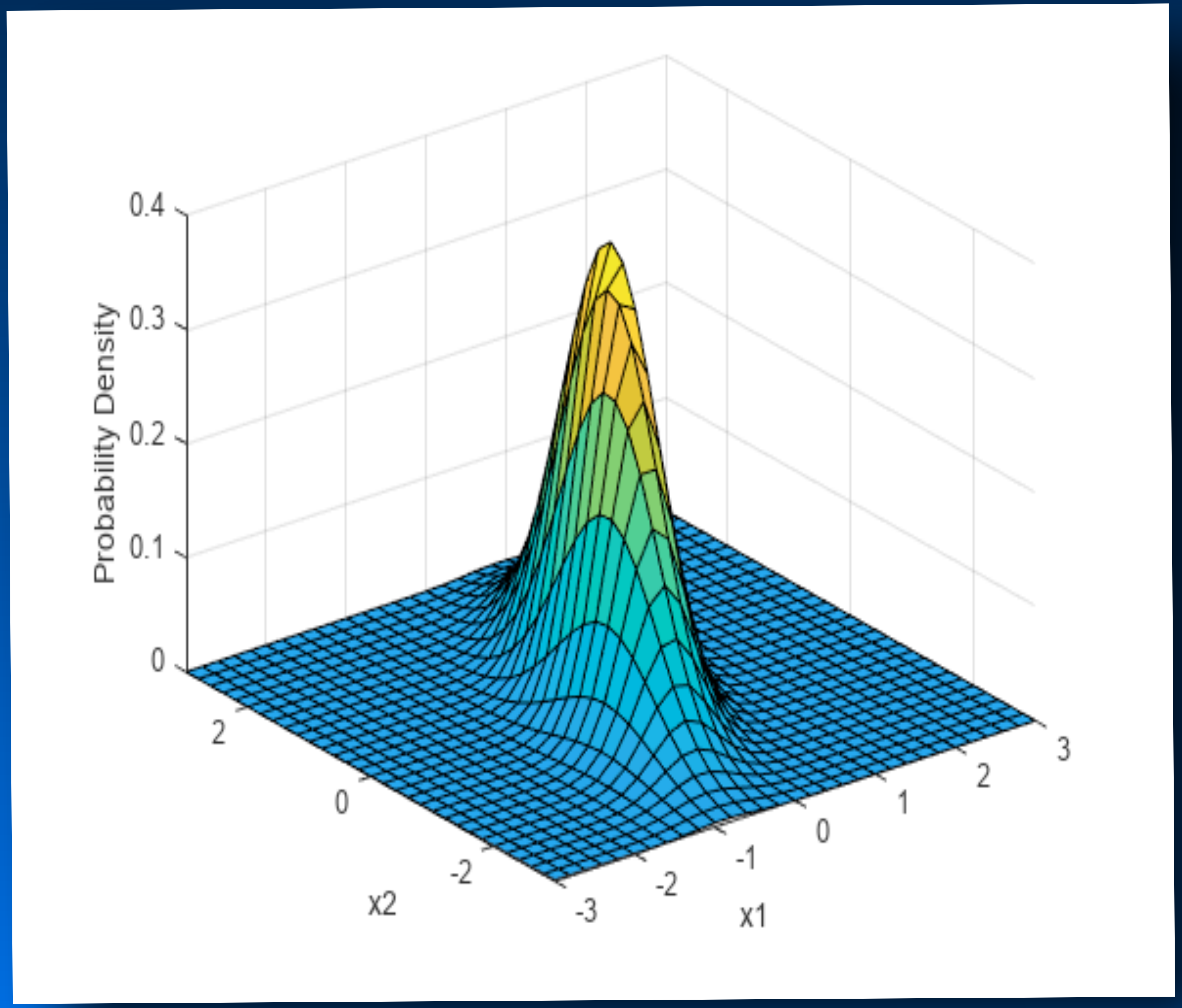


$$s^2 = \frac{n \cdot \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n \cdot (n-1)}$$

Where: x = variable value;
n = sample size.

Statistics as Analysis

Covariance - the joint variation of two variables about their common mean.



$$COV_{jk} = \frac{\sum_{i=1}^{n} x_{ij} \cdot x_{ik} - (\sum_{i=1}^{n} x_{ij} \cdot \sum_{i=1}^{n} x_{ik})}{n \cdot (n-1)}$$

Where: x = variable value;
n = sample size.

Matrices as Data Storage

Covariance Matrix (between columns)

		Variables									
		1	2	3	4	5	6	7	8	9	10
Variables	1	570.70	145.85	214.20	689.40	28.90	-213.80	-90.75	-18.85	227.35	-232.80
	2	145.85	587.30	60.60	237.20	511.95	29.85	144.50	-92.55	101.80	165.10
	3	214.20	60.60	991.20	288.90	-197.10	358.70	209.00	666.40	771.10	468.20
	4	689.40	237.20	288.90	857.80	92.30	-269.85	-79.00	31.80	241.20	-154.35
	5	28.90	511.95	-197.10	92.30	518.30	-52.85	94.75	-218.95	-138.05	105.15
	6	-213.80	29.85	358.70	-269.85	-52.85	375.20	167.25	231.65	374.10	216.95
	7	-90.75	144.50	209.00	-79.00	94.75	167.25	130.50	165.25	144.00	272.50
	8	-18.85	-92.55	666.40	31.80	-218.95	231.65	165.25	618.30	344.45	587.90
	9	227.35	101.80	771.10	241.20	-138.05	374.10	144.00	344.45	821.30	47.35
	10	-232.80	165.10	468.20	-154.35	105.15	216.95	272.50	587.90	47.35	931.20

Covariance Matrix = A square symmetric matrix.

Matrices as Data Storage

Covariance Matrix (Between Columns)

		Variables									
		1	2	3	4	5	6	7	8	9	10
Variables	1	570.70	145.85	214.20	689.40	28.90	-213.80	-90.75	-18.85	227.35	-232.80
	2	145.85	587.30	60.60	237.20	511.95	29.85	144.50	-92.55	101.80	165.10
	3	214.20	60.60	991.20	288.90	-197.10	358.70	209.00	666.40	771.10	468.20
	4	689.40	237.20	288.90	857.80	92.30	-269.85	-79.00	31.80	241.20	-154.35
	5	28.90	511.95	-197.10	92.30	518.30	-52.85	94.75	-218.95	-138.05	105.15
	6	-213.80	29.85	358.70	-269.85	-52.85	375.20	167.25	231.65	374.10	216.95
	7	-90.75	144.50	209.00	-79.00	94.75	167.25	130.50	165.25	144.00	272.50
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	9	227.35	101.80	771.10	241.20	-138.05	374.10	144.00	344.45	821.30	47.35
	10	-232.80	165.10	468.20	-154.35	105.15	216.95	272.50	587.90	47.35	931.20

Trace = Covariance of a variable with itself = variance.

Matrices as Data Storage

Covariance Matrix (between columns)

		Variables									
		1	2	3	4	5	6	7	8	9	10
Variables	1	570.70	145.85	214.20	689.40	28.90	-213.80	-90.75	-18.85	227.35	-232.80
	2	145.85	587.30	60.60	237.20	511.95	29.85	144.50	-92.55	101.80	165.10
	3	214.20	60.60	991.20	288.90	-197.10	358.70	209.00	666.40	771.10	468.20
	4	689.40	237.20	288.90	857.80	92.30	-269.85	-79.00	31.80	241.20	-154.35
	5	28.90	511.95	-197.10	92.30	518.30	-52.85	94.75	-218.95	-138.05	105.15
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	7	-90.75	144.50	209.00	-79.00	94.75	167.25	130.50	165.25	144.00	272.50
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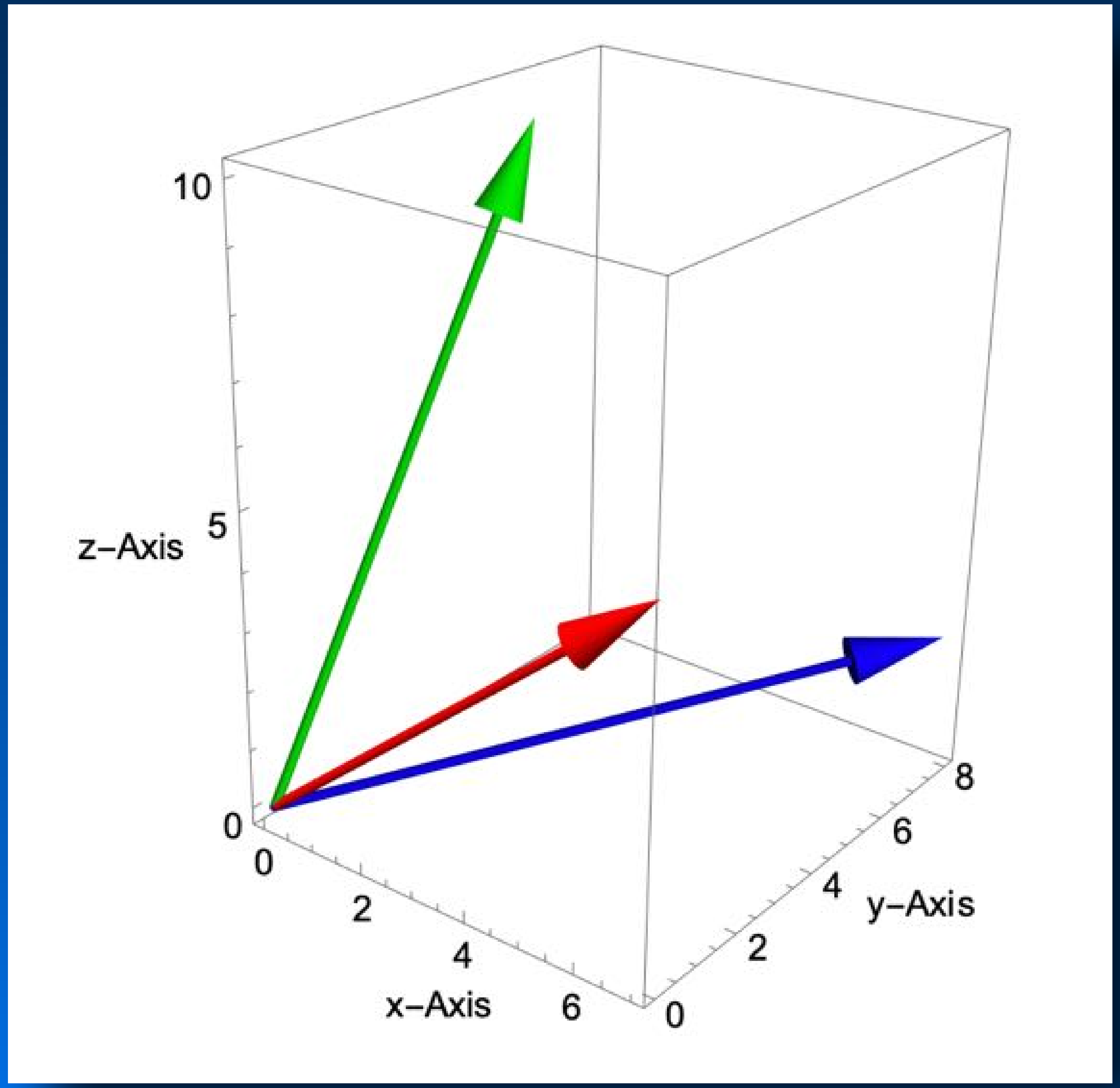
	10	-232.80	165.10	468.20	-154.35	105.15	216.95	272.50	587.90	47.35	931.20

Diagonals express the geometric relations (orientations) between variables.

Statistics as Analysis

The distance between sets of objects can be visualized quantified by regarding each object as a vector and calculating the distance between the vector heads.



Euclidean Distance

$$d_{jk} = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2}$$

Matrices as Data Storage

Distance Matrix (Cetween Rows)

		Objects									
		1	2	3	4	5	6	7	8	9	10
Objects	1	0.00	145.85	214.20	689.40	28.90	213.80	90.75	18.85	227.35	232.80
	2	145.85	0.00	60.60	237.20	511.95	29.85	144.50	92.55	101.80	165.10
	3	214.20	60.60	0.00	288.90	-97.10	358.70	209.00	666.40	771.10	468.20
	4	689.40	237.20	288.90	0.00	92.30	269.85	79.00	31.80	241.20	154.35
	5	28.90	511.95	197.10	92.30	0.00	52.85	94.75	218.95	138.05	105.15
	6	213.80	29.85	358.70	269.85	52.85	0.00	167.25	231.65	374.10	216.95
	7	90.75	144.50	209.00	79.00	94.75	167.25	0.00	165.25	144.00	272.50
	8	18.85	92.55	666.40	31.80	218.95	231.65	165.25	0.00	344.45	587.90
	9	227.35	101.80	771.10	241.20	138.05	374.10	144.00	344.45	0.00	47.35
	10	232.80	165.10	468.20	154.35	105.15	216.95	272.50	587.90	47.35	0.00

Distance Matrix = A Square Symmetric Matrix

Matrices as Data Storage

Distance Matrix (Cetween Rows)

		Objects									
		1	2	3	4	5	6	7	8	9	10
Objects	1	0.00	145.85	214.20	689.40	28.90	213.80	90.75	18.85	227.35	232.80
	2	145.85	0.00	60.60	237.20	511.95	29.85	144.50	92.55	101.80	165.10
	3	214.20	60.60	0.00	288.90	-97.10	358.70	209.00	666.40	771.10	468.20
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	5	28.90	511.95	197.10	92.30	0.00	52.85	94.75	218.95	138.05	105.15
	6	213.80	29.85	358.70	269.85	52.85	0.00	167.25	231.65	374.10	216.95
	7	90.75	144.50	209.00	79.00	94.75	167.25	0.00	165.25	144.00	272.50
	8	18.85	92.55	666.40	31.80	218.95	231.65	165.25	0.00	344.45	587.90
	9	227.35	101.80	771.10	241.20	138.05	374.10	144.00	344.45	0.00	47.35
	10	232.80	165.10	468.20	154.35	105.15	216.95	272.50	587.90	47.35	0.00

Trace = Distance between an object and Itself = 0.00

Matrices as Data Storage

Distance Matrix (Cetween Rows)

		Objects									
		1	2	3	4	5	6	7	8	9	10
Objects	1	0.00	145.85	214.20	689.40	28.90	213.80	90.75	18.85	227.35	232.80
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	5	28.90	511.95	197.10	92.30	0.00	52.85	94.75	218.95	138.05	105.15
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	9	227.35	101.80	771.10	241.20	138.05	374.10	144.00	344.45	0.00	47.35
	10	232.80	165.10	468.20	154.35	105.15	216.95	272.50	587.90	47.35	0.00

Diagonals always contain uniformly positive values.

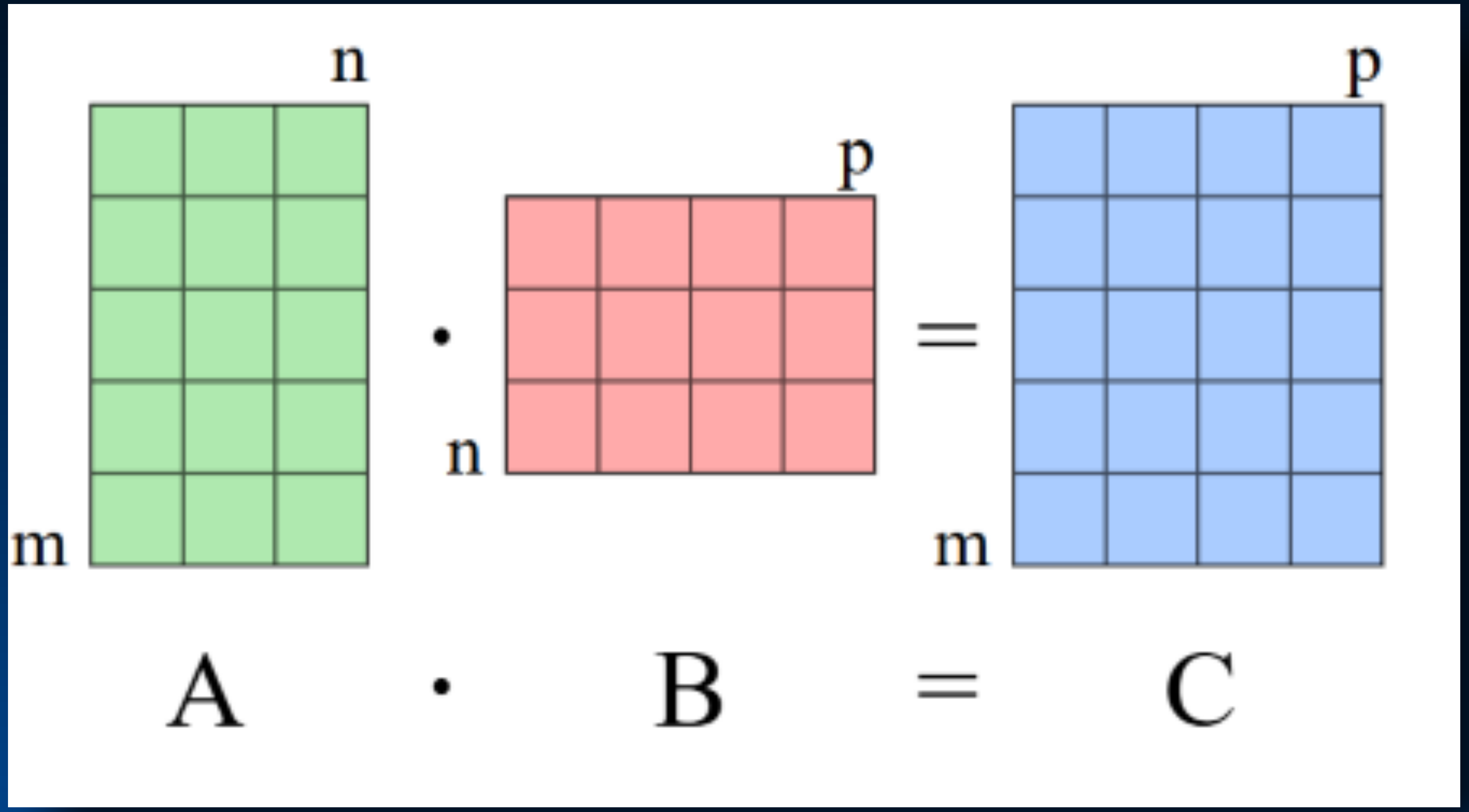
Matrix Notation

In mathematics we use a special notation to represent the structure and values assigned to matrix cells.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The first subscript refers to the row number, the second subscript refers to the column number, 3D matrix cells have three subscripts, 4D matrix cells have four subscripts, etc.

Matrix Arithmetic



Matrix arithmetic is a collection of procedures that can be used to transform, combine and/or partition matrices according to the basic rules of arithmetic. Since matrices are collections of information, matrix arithmetic can often be used to perform complex calculations in a more computationally efficient manner than the standard list-based formulations. Many descriptive statistical parameters also have matrix arithmetic formula. Most spreadsheet and mathematical-calculation software applications have built-in routines to perform matrix arithmetic calculations

Matrix Notation

In mathematics we use a special notation to represent the structure and values assigned to matrix cells.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The first subscript refers to the row number, the second subscript refers to the column number, 3D matrix cells have three subscripts, 4D matrix cells have four subscripts, etc.

Matrix Notation

In mathematics we use a special notation to represent the structure and values assigned to matrix cells.

$$A = \begin{pmatrix} 5 & 3 & 4 \\ 1 & 6 & 10 \\ 7 & 8 & 2 \end{pmatrix}$$

What is the cell number of value 8?

Matrix Arithmetic

Transposition - The transposition operation exchanges the rows and columns of a matrix without changing the relative order of the cells.

$$A = \begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} \qquad A^T = \begin{pmatrix} 5 & 1 \\ 3 & 5 \\ 4 & 10 \end{pmatrix}$$

Note how the matrix-cell subscripts are changed by the transposition transformation. Computer spreadsheets (which are, essentially nothing more than interactive matrices) have built-in transposition routines

Matrix Arithmetic

Matrix Addition - Two matrices of equivalent size and dimension can be added together by adding the values of equivalent cells.

$$A + B = C$$

$$\begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} + \begin{pmatrix} 8 & 5 & 1 \\ 6 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 8 & 5 \\ 7 & 8 & 12 \end{pmatrix}$$

Note matrices must have exactly the same sizes and shapes in order to be added together.

Matrix Arithmetic

Matrix Subtraction - Two matrices of equivalent size and dimension can be subtracted by subtracting the values of equivalent cells.

$$A - B = C$$

$$\begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} - \begin{pmatrix} 8 & 5 & 1 \\ 6 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 3 \\ -5 & 2 & 8 \end{pmatrix}$$

Note matrices must have exactly the same sizes and shapes in order to be subtracted together.

Matrix Arithmetic

Scalar Multiplication - Any matrix can be scaled by a constant by multiplying the value of each cell by the constant.

$$\alpha \cdot A = B$$

$$3 \cdot \begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} = \begin{pmatrix} 15 & 9 & 12 \\ 3 & 15 & 30 \end{pmatrix}$$

Note matrices of any size and shape can be scaled by scalar multiplication.

Matrix Arithmetic

Scalar Division - Any matrix can be reduced by a constant by dividing the value of each cell by the constant.

$$\alpha \div A = B$$

$$3 \div \begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} = \begin{pmatrix} 0.6 & 1.0 & 0.75 \\ 3.0 & 0.6 & 0.3 \end{pmatrix}$$

Note matrices of any size and shape can be reduced by scalar division.

Matrix Arithmetic

Matrix Multiplication - This operation is restricted to matrices of corresponding dimensions. Here A (a 3 x 2 matrix) and B (a 2 x 3 matrix) have the dimension 3 in common and in opposing orientations. These matrixes can be multiplied together by using a somewhat complicated pattern of internal products and sums. The result is a 3 x 3 matrix.

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \quad A \cdot B = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

Matrix Multiplication (Example)

$$B = \begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 8 & 9 \\ 10 & 5 \\ 3 & 6 \end{pmatrix} \qquad A \cdot B = \begin{pmatrix} (8 \cdot 5) + (9 \cdot 1) & (8 \cdot 3) + (9 \cdot 5) & (8 \cdot 4) + (9 \cdot 10) \\ (10 \cdot 5) + (5 \cdot 1) & (10 \cdot 3) + (5 \cdot 5) & (10 \cdot 4) + (5 \cdot 10) \\ (3 \cdot 5) + (6 \cdot 1) & (3 \cdot 3) + (6 \cdot 5) & (3 \cdot 4) + (6 \cdot 10) \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 40 + 9 & 24 + 45 & 32 + 90 \\ 50 + 5 & 30 + 25 & 40 + 50 \\ 15 + 6 & 9 + 30 & 12 + 60 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 49 & 69 & 122 \\ 55 & 55 & 90 \\ 21 & 39 & 72 \end{pmatrix}$$

Matrix Arithmetic

Matrix Inversion - This operation is used to facilitate matrix division by square matrices in the sense that regular division is simply multiplication by the divisor's reciprocal.

$$A = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \quad A' = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 0.2 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 0.2 \end{pmatrix}$$

$$A \cdot A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

An inverted matrix will be exactly the same size and shape as the original matrix. However, not all matrices can be inverted.

Matrix Inversion (Example)

$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Place the matrix to be inverted next to the identify matrix.
$\begin{pmatrix} 1 & 2.5 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} 0.25 & 0 \\ 0 & 1 \end{pmatrix}$	Divide row 1 of both matrices by 4.
$\begin{pmatrix} 1 & 2.5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}$	Divide row 2 by 10.
$\begin{pmatrix} 1 & 2.5 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.25 & 0 \\ -0.25 & 0.1 \end{pmatrix}$	Subtract row 1 from row 2.
$\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1.5 & -0.5 \\ -0.25 & 0.1 \end{pmatrix}$	Multiply row 2 by 5 and subtract from row 1.
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$	Divide row 2 by 0.5.

Matrix Determinate

A matrix's determinant is a scalar value calculated from a square matrix that represents the scaling factor of a linear transformation. Mathematically, the determinate has many uses (e.g., it can be used to determine whether a matrix can be inverted, to solve systems of equations, to find eigenvalues). The determinate is computed by specific formulas based on matrix size.

$$|A| = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad) - (bc)$$

Matrix Inversion & Determinants

- A determinant can be calculated for any matrix.
- Determinants can have positive values, negative values or zero.
- Asymmetric matrices can have determinant of any value.
- Symmetric matrices can only have positive or zero determinant values.
- If a matrix has a positive determinant it can be inverted and is said to be “non-singular”.
- If a matrix has a determinant value of zero (0.0) it cannot be inverted using normal procedures and is said to be “singular”.
- Methods are available to invert singular matrices (e.g., pseudoinverse).

Basic Matrix Operations

Minor Product Moment $(A'B)$ - This operation is restricted to two matrices of equal order, one a column vector (= the pre-factor, A) and the other a row vector (the post-factor, B). When these two are multiplied their product is a scalar value that represents the sum of products of the corresponding elements.

$A'B$ = Minor Product Moment

$$B = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

$$A' = \begin{pmatrix} 2 & 4 & 1 \end{pmatrix} \quad 15$$

A	B	Prod.
2	3	6
4	1	4
1	5	5
	Σ	15

Basic Matrix Operations

Major Product Moment (AB') - This operation is restricted to two matrices of equal order, in this case one a column vector (= the pre-factor, A) and the other a row vector (the post-factor, B). When these two are multiplied the product is a matrix of products between the corresponding elements.

AB' = Major Product Moment

$$B' = \begin{pmatrix} 6 & 4 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \qquad AB' = \begin{pmatrix} 6 & 2 & 10 \\ 12 & 4 & 20 \\ 3 & 1 & 5 \end{pmatrix}$$

A	B	Prod.
2	3	6
4	1	4
1	5	5
	Σ	15

Basic Matrix Operations

Vector Length - This can also be calculated as the square root of a vector premultiplied by its transpose (= root of the minor product moment of the vector with itself).

$$\text{Length} = \sqrt{A' \cdot A}$$

$$A = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$A' = (6 \quad 4) \qquad 52$$

$$\text{Length} = \sqrt{52} = 7.2111$$

Point	x	y
Begin	2	3
End	8	7
Δ	6	4

Basic Matrix Operations

Angle Between Two Vectors - This can also be calculated as the ratio of the minor product of the two vectors divided by the product of their lengths.

$$\text{Angle} = \frac{A'B}{\sqrt{A'A} \cdot \sqrt{B'B}}$$

$$A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

$$A'B = 15$$

$$\sqrt{A'A} = 4.58$$

$$\sqrt{B'B} = 5.92$$

$$\text{Cosine}_{\text{Radians}} = 0.984$$

$$\text{Angle} = 56.41^\circ$$

To calculate the angle between a vector and any coordinate axis make the second vector coincide with the axis.

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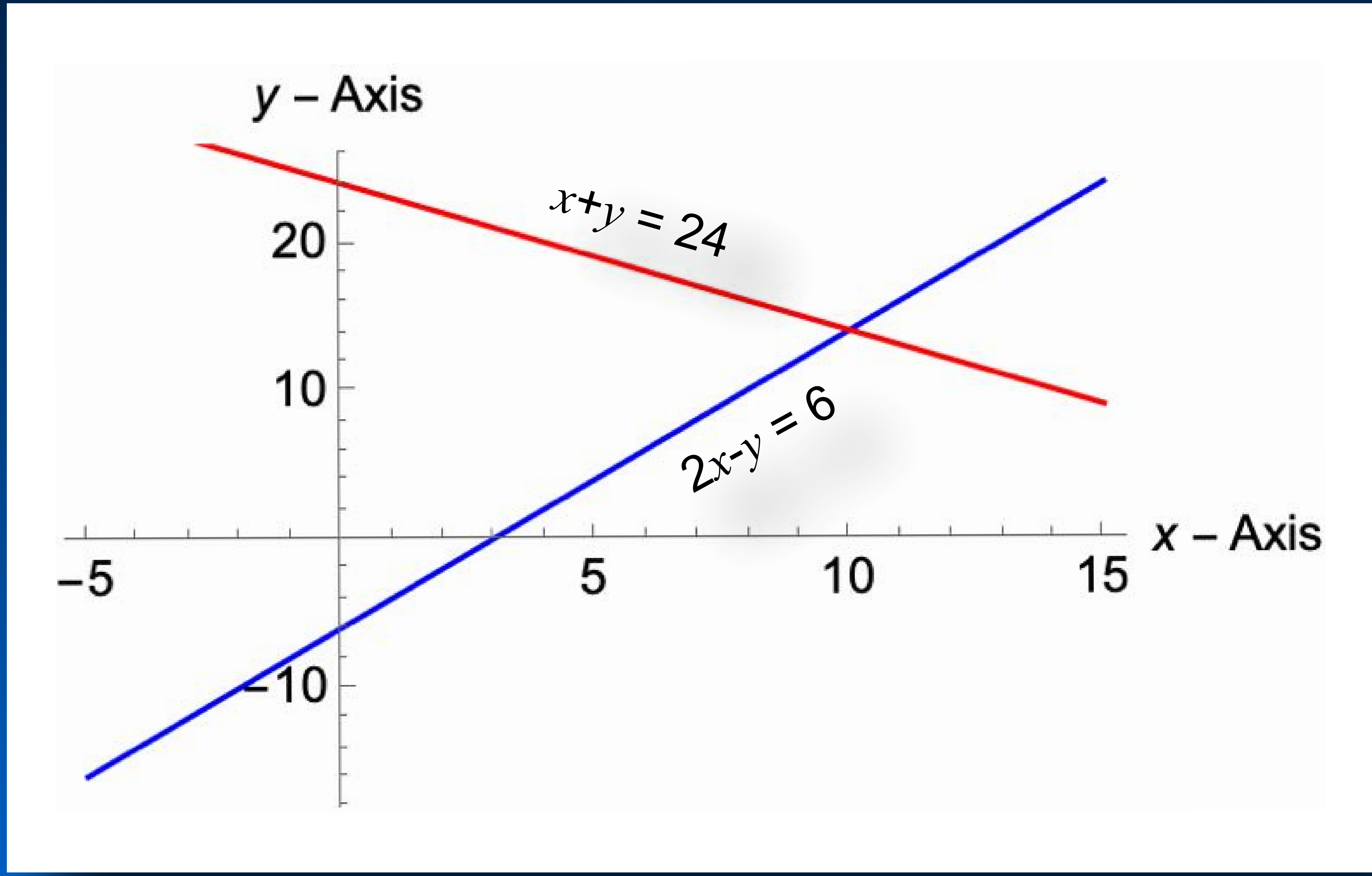
To calculate the angle between a vector and any coordinate axis make the second vector coincide with the axis.

Basic Matrix Operations

Solving Simultaneous Equations - Geometrically the intersection point of lines can be found by setting their equations equal to one another and find the unique coord-innate points common to each. This can be carries out very ef-ficiently using matrix methods.



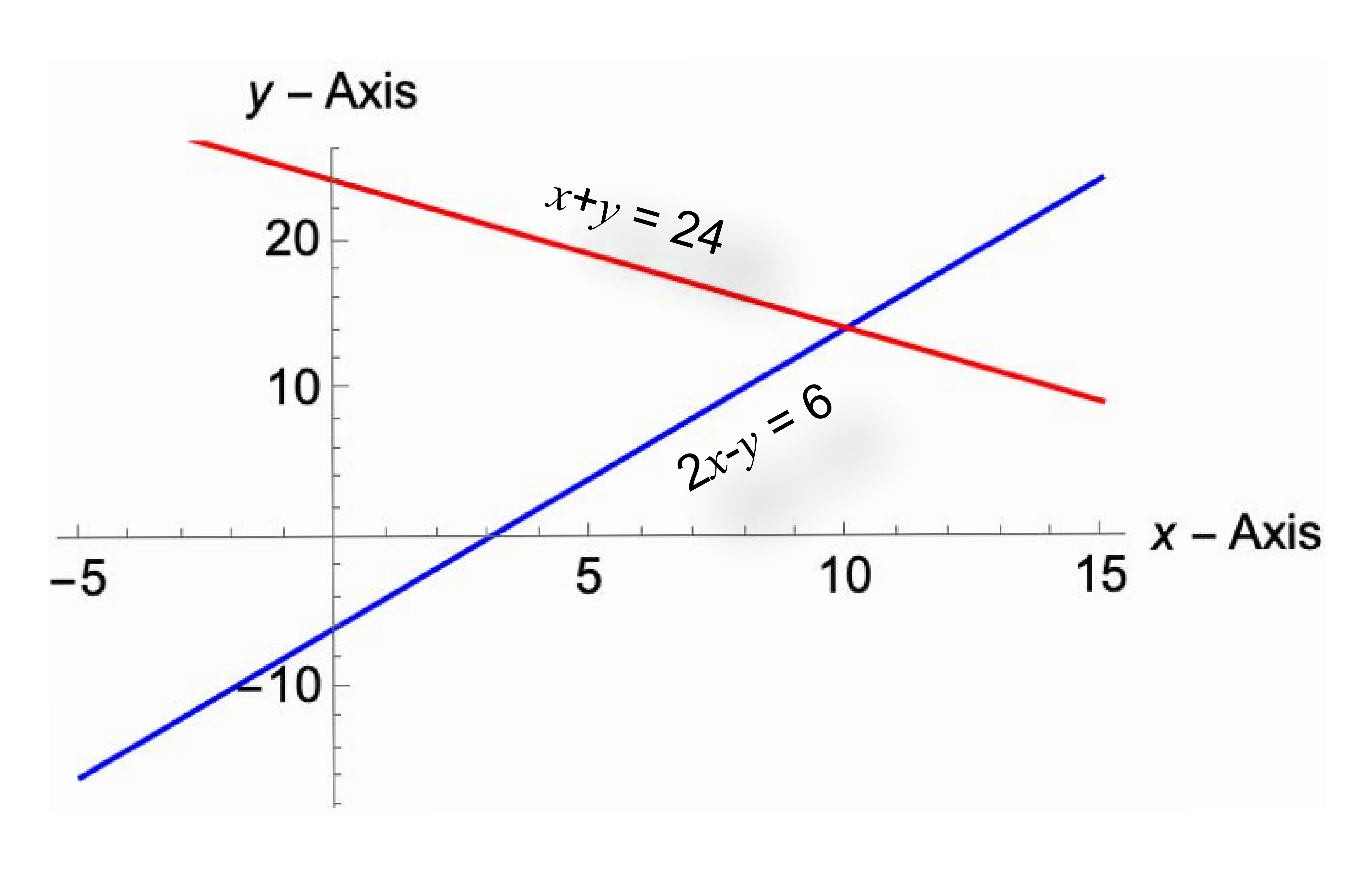
$$2x - y = 6$$

$$x + y = 24 \qquad A \cdot X = B$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \end{pmatrix}$$

Basic Matrix Operations

Solving Simultaneous Equations - Geometrically the intersection point of lines can be found by setting their equations equal to one another and find the unique coord-innate points common to each. This can be carries out very ef-ficiently using matrix methods.



$$2x - y = 6$$

$$x + y = 24 \qquad B \cdot X = A$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 \\ 24 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

Matrix Transformations

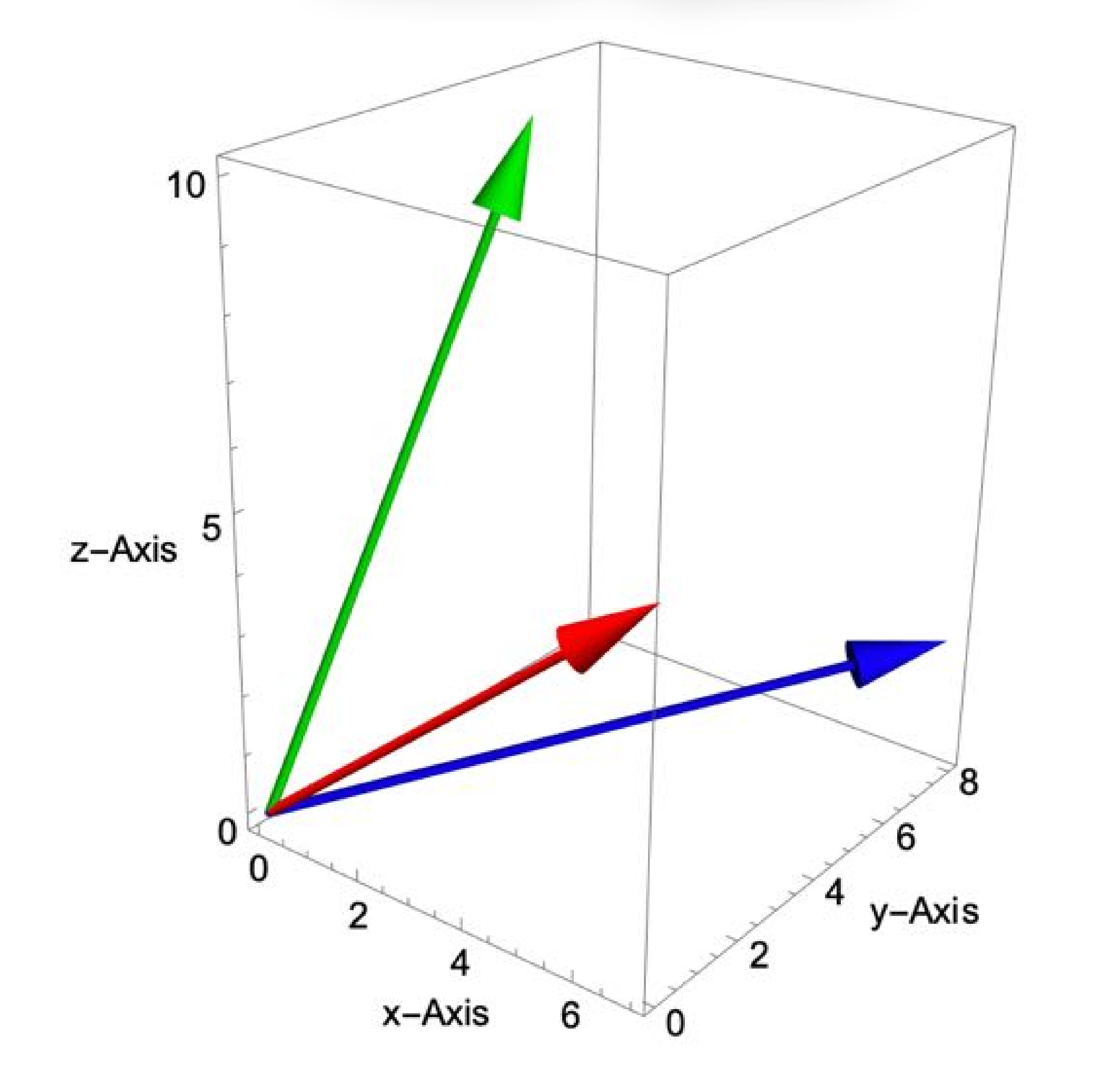
Matrix Transformations

Matrices can be used to transform arrays of numerical values into images – in this case, vectors – or to transform images into numerical values.

Matrix Representation

$$A = \begin{pmatrix} 5 & 3 & 4 \\ 1 & 5 & 10 \\ 7 & 8 & 2 \end{pmatrix} \longleftrightarrow$$

Vector-Space



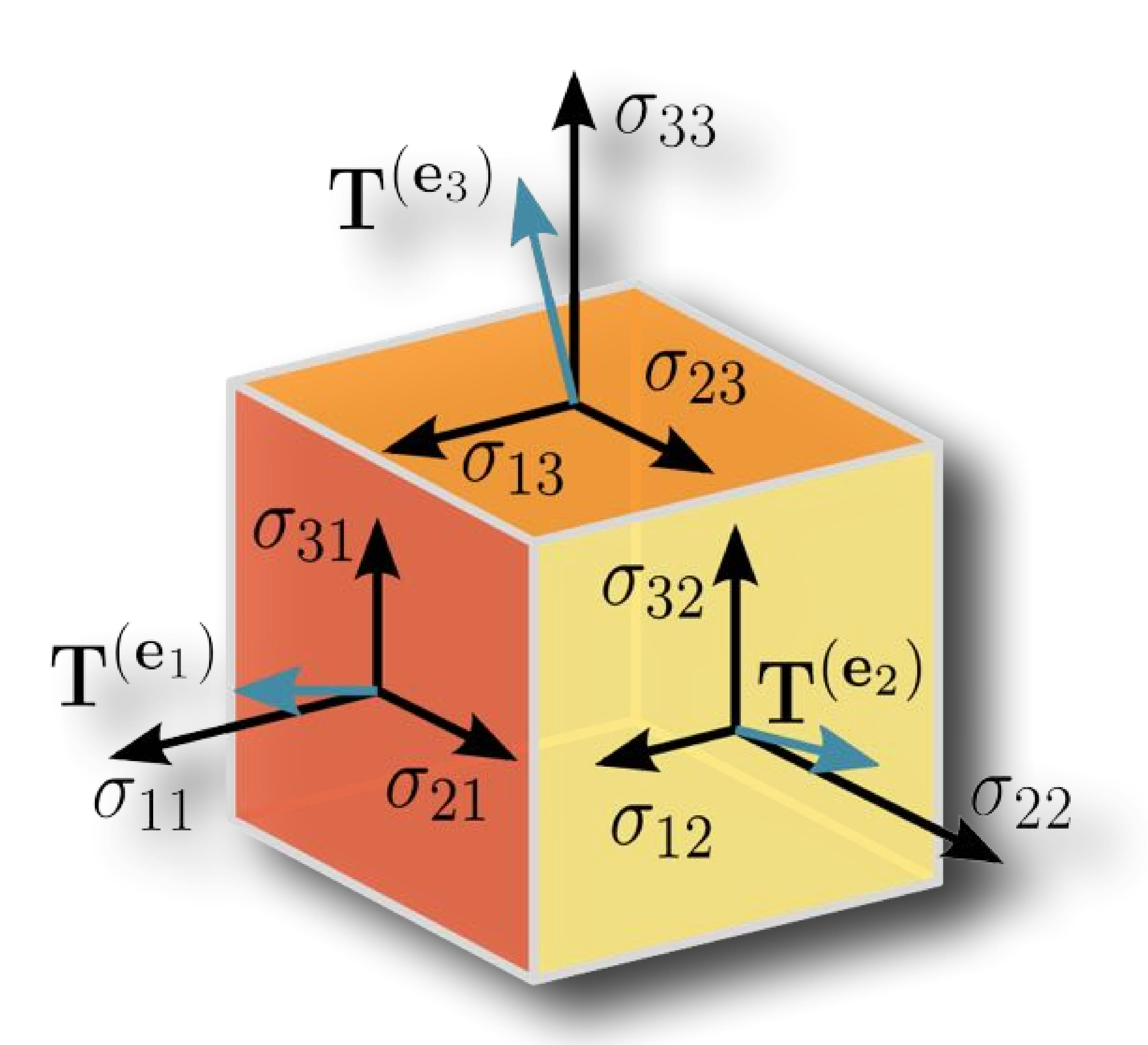
Matrices & Tensors

Tensor - an algebraic object that describes a linear mapping of one set of algebraic objects to another. Such mappings are usually (but not exclusively) expressed in terms of scalars and vectors.

The second-order Cauchy stress tensor in the basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$): $\mathbf{T} = \left[\mathbf{T}^{(\mathbf{e}_1)} \mathbf{T}^{(\mathbf{e}_2)} \mathbf{T}^{(\mathbf{e}_3)}\right]$, or

$$\mathbf{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

The columns are the respective stress vectors that act in the center of the cube, regarding to planes orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. When $\mathbf{v}$ is given in this basis, the product of the two tensors, $\mathbf{T} \cdot \mathbf{v}$, performed as matrix multiplication, yields the stress vector $\mathbf{T}^{(\mathbf{v})}$ in that point, which has its shear part in the plane orthogonal to $\mathbf{v}$.

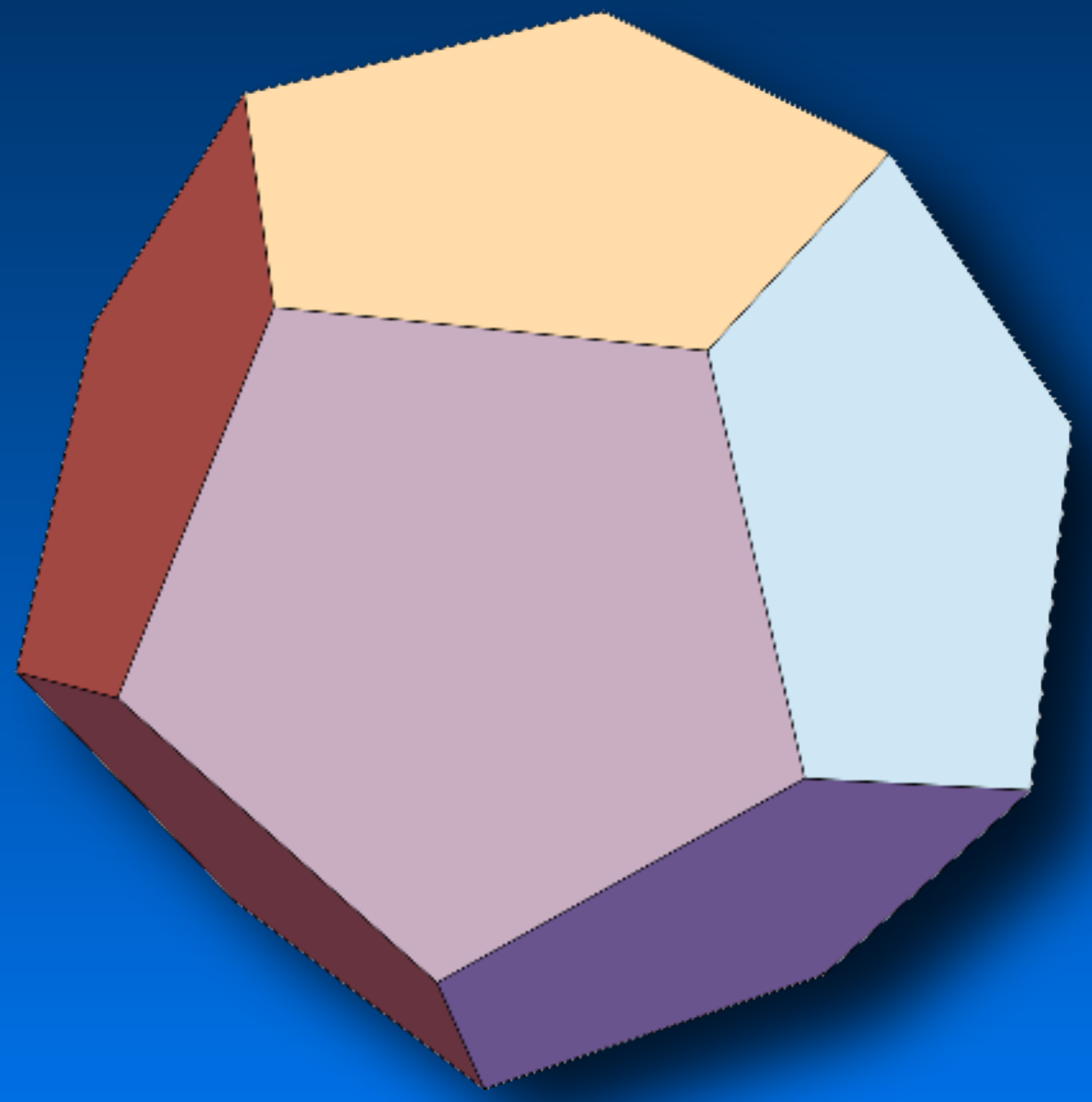


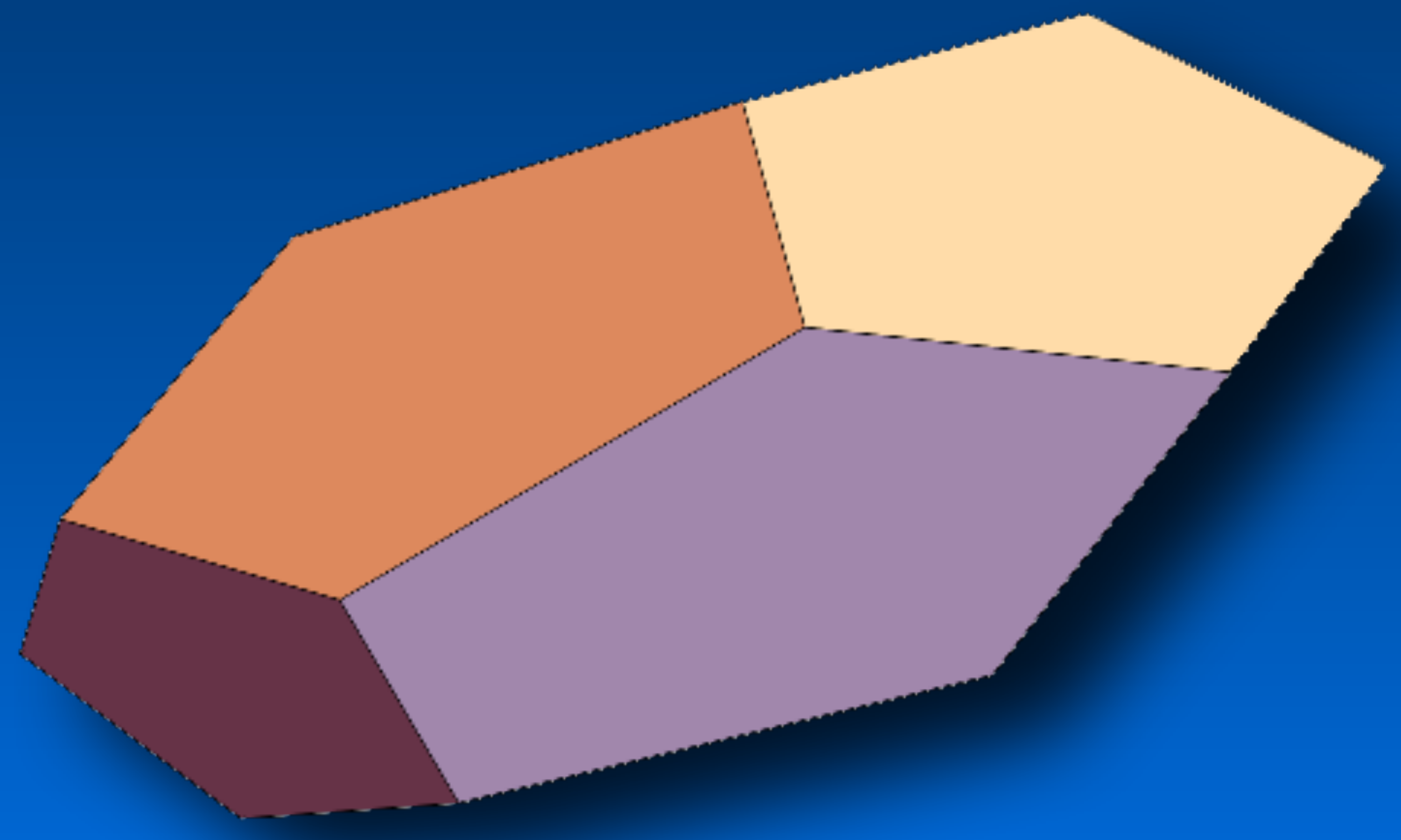
Matrix Transformations

Linear (Affine) Transformations

Shearing Transformation

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



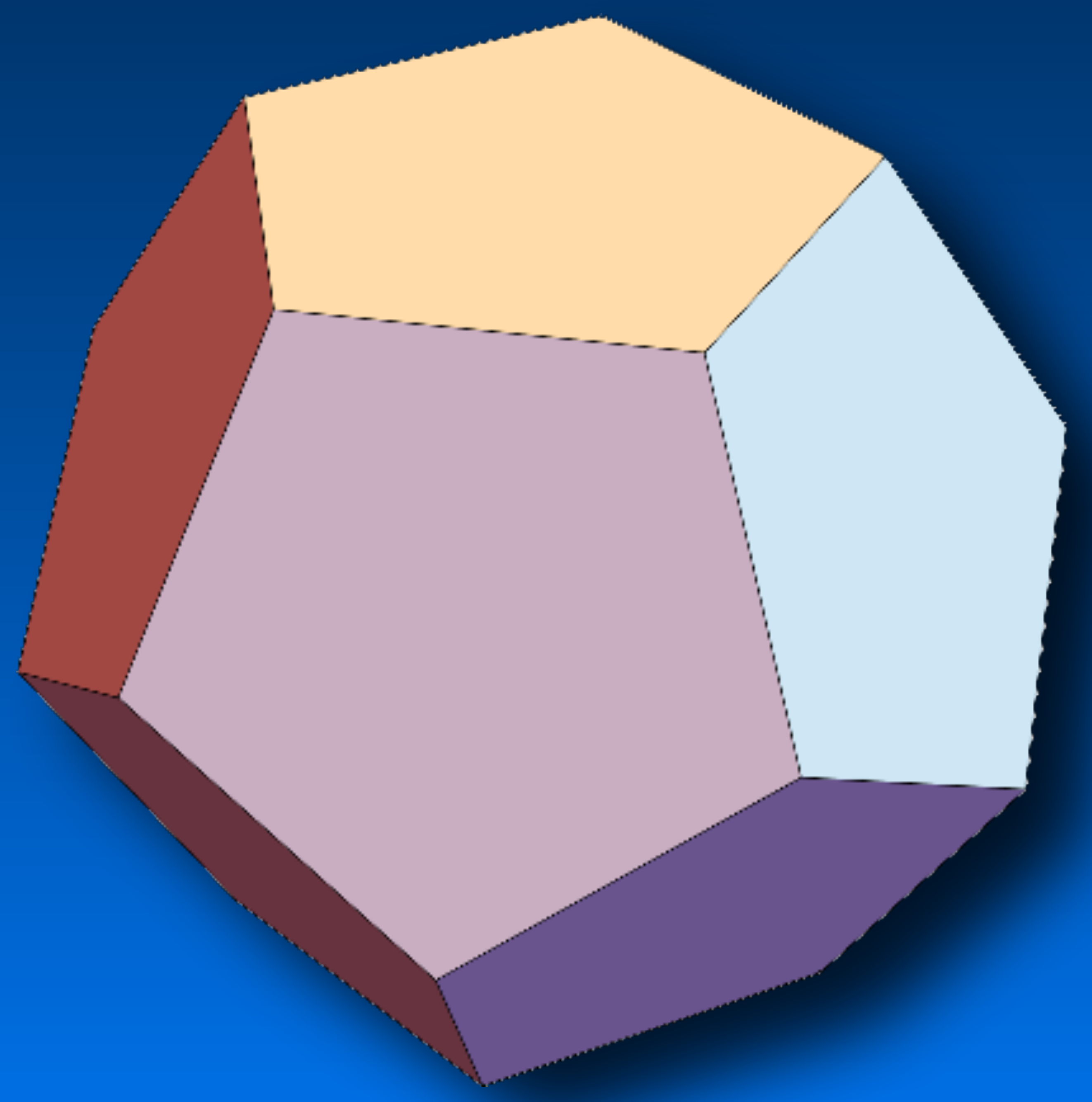


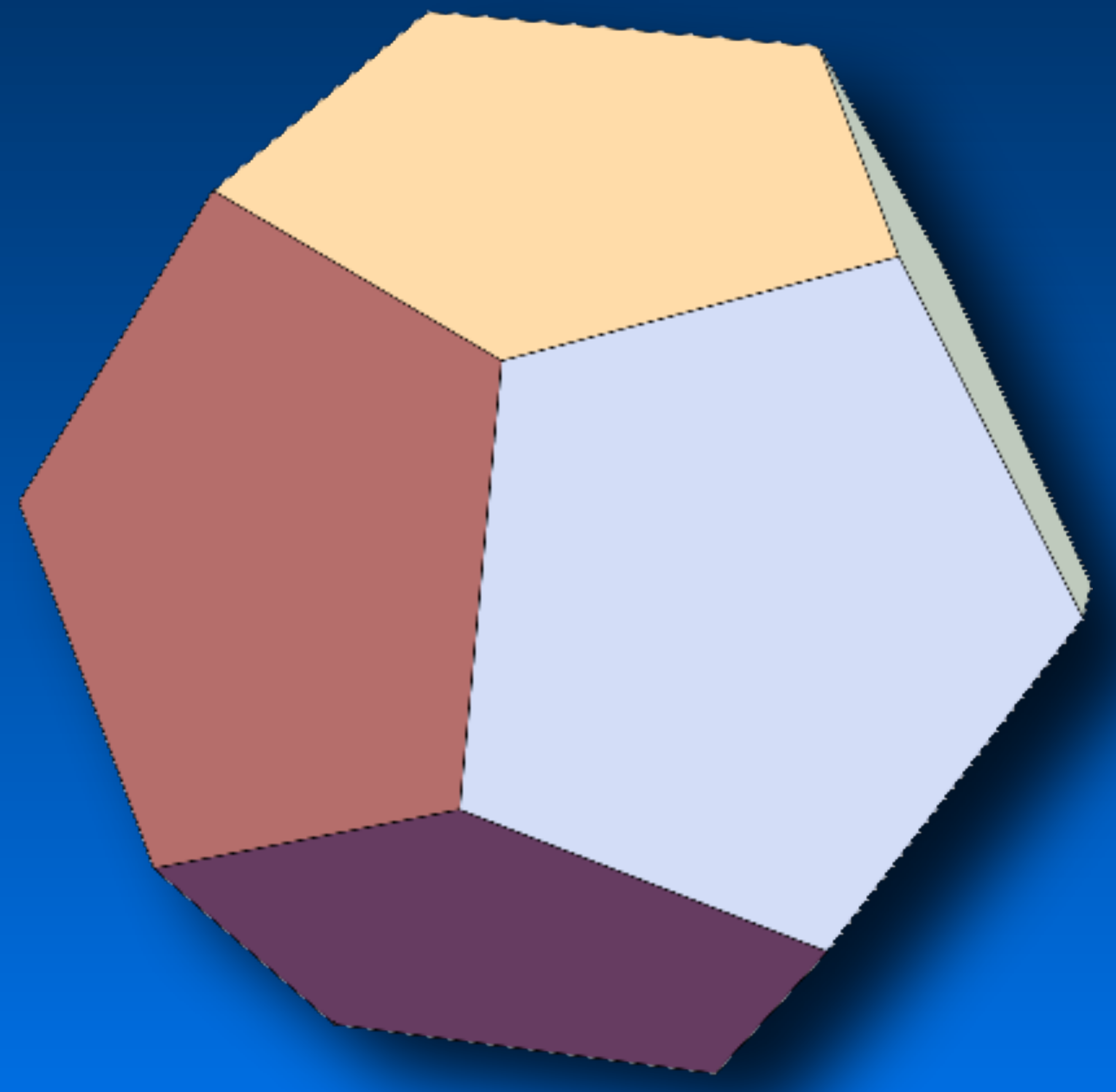
Matrix Transformations

Linear (Affine) Transformations

Reflection Transformation

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

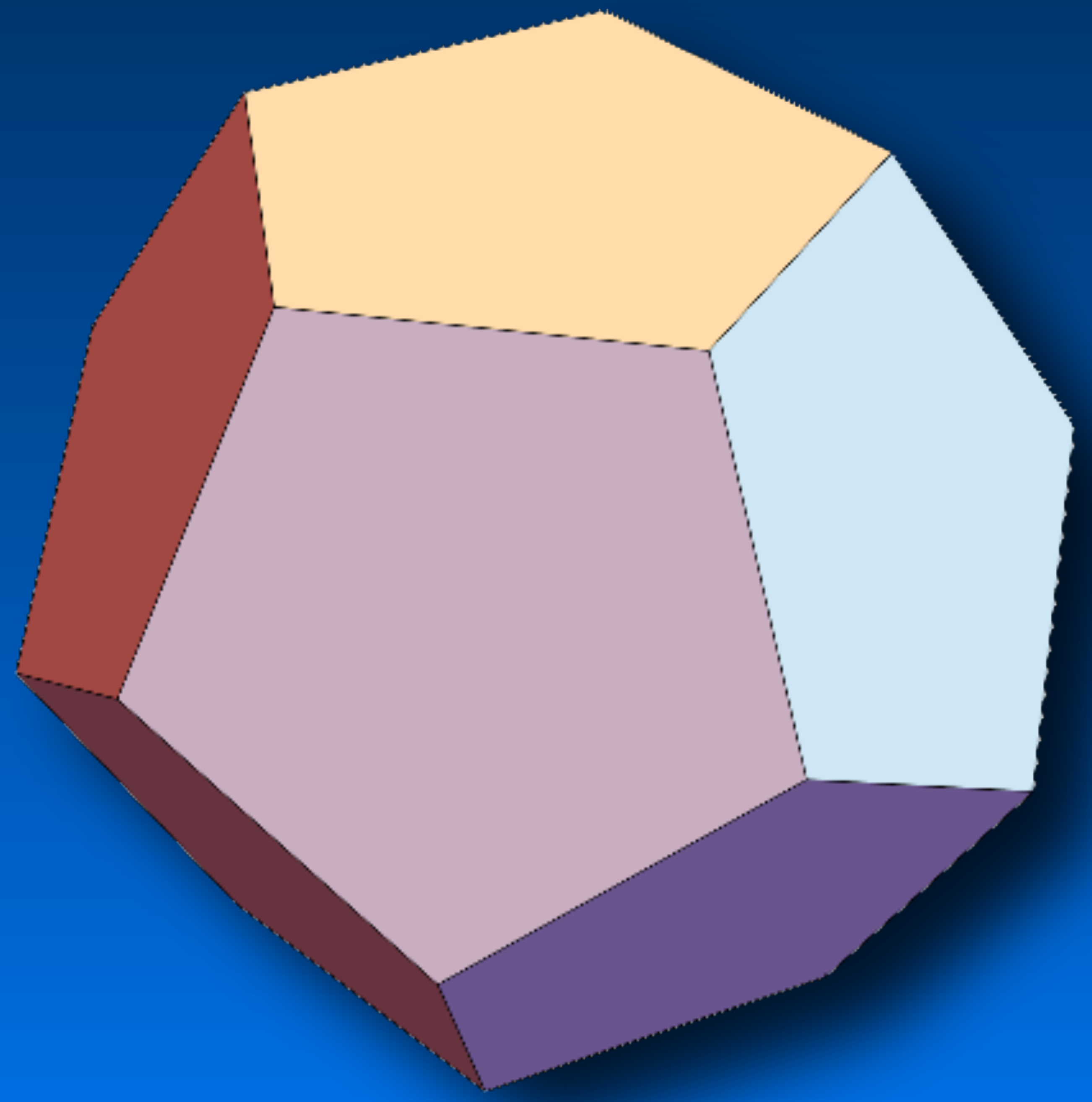




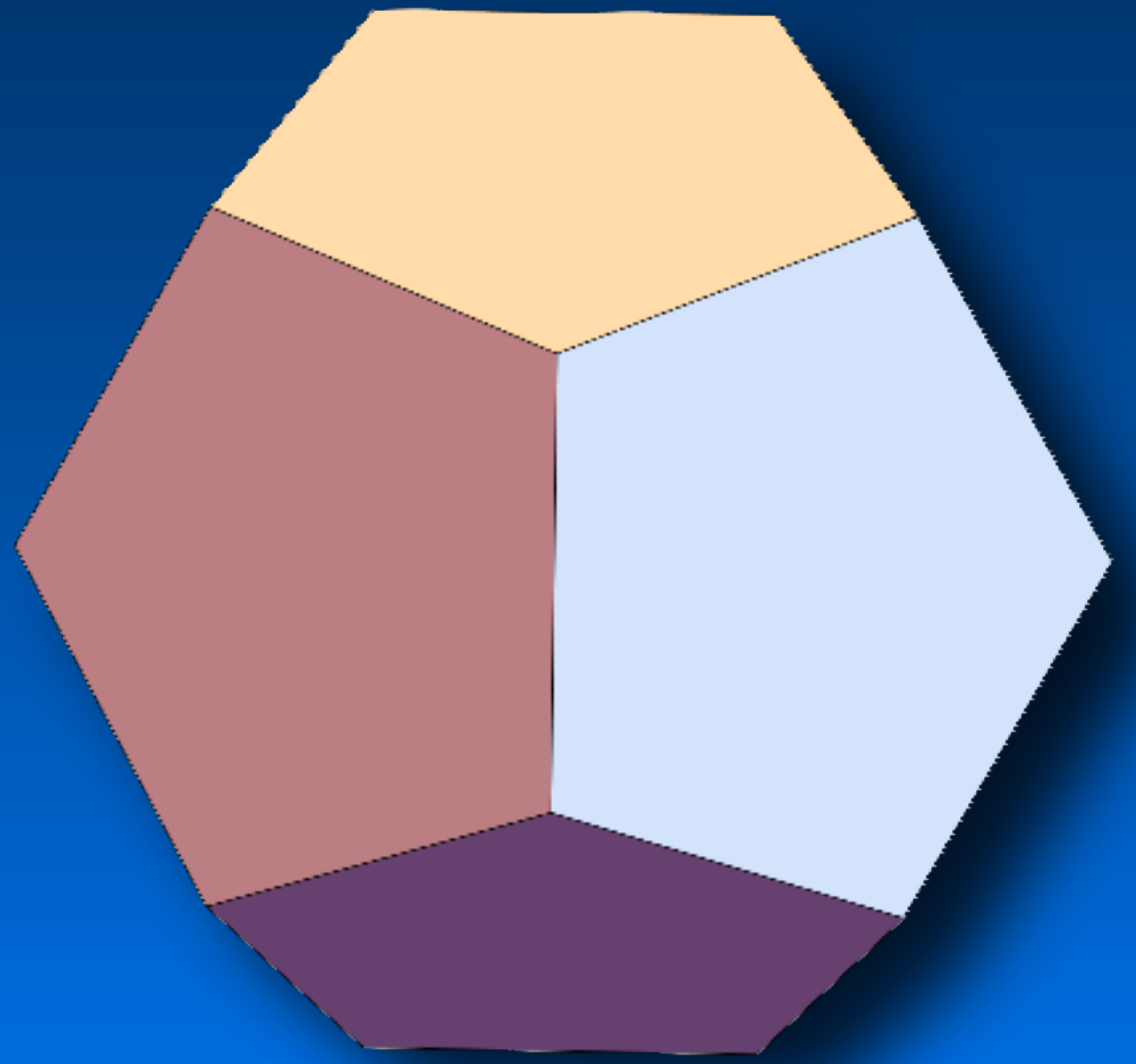
Matrix Transformations

Linear (Affine) Transformations

Rotation Transformation



$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

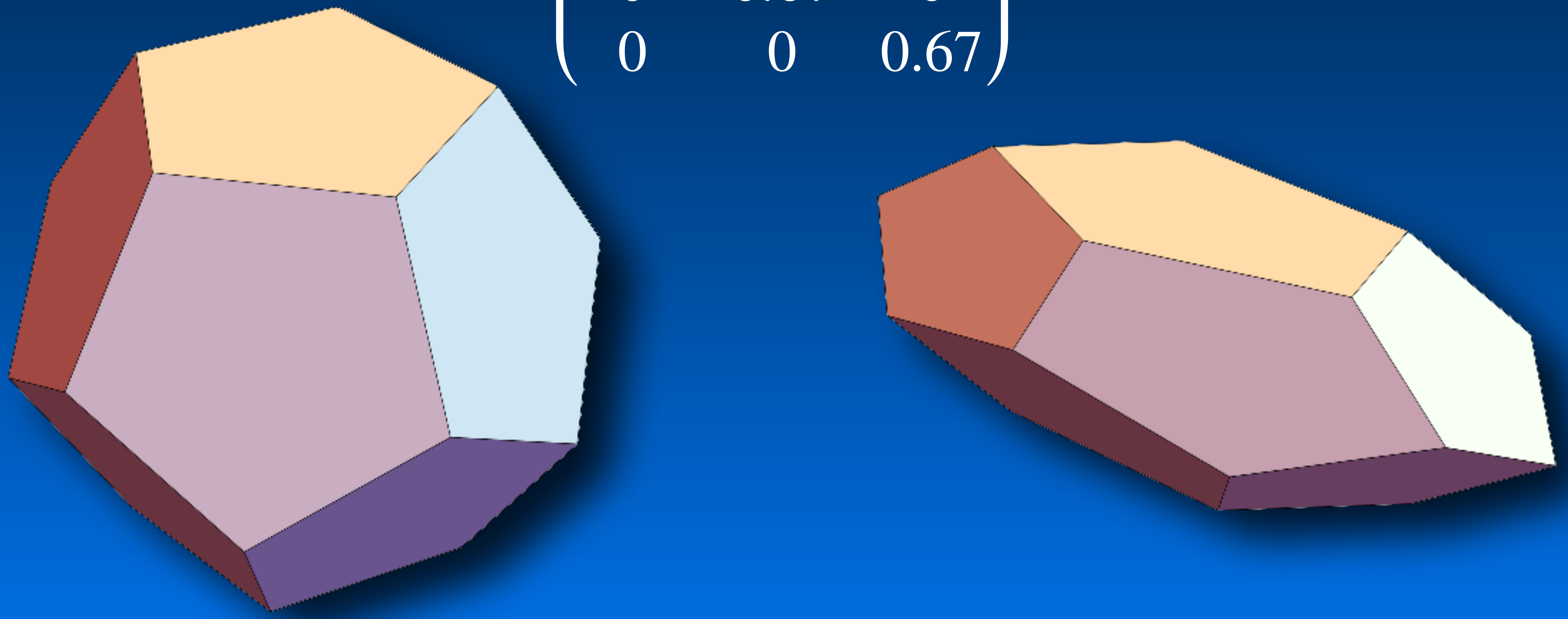


Matrix Transformations

Linear (Affine) Transformations

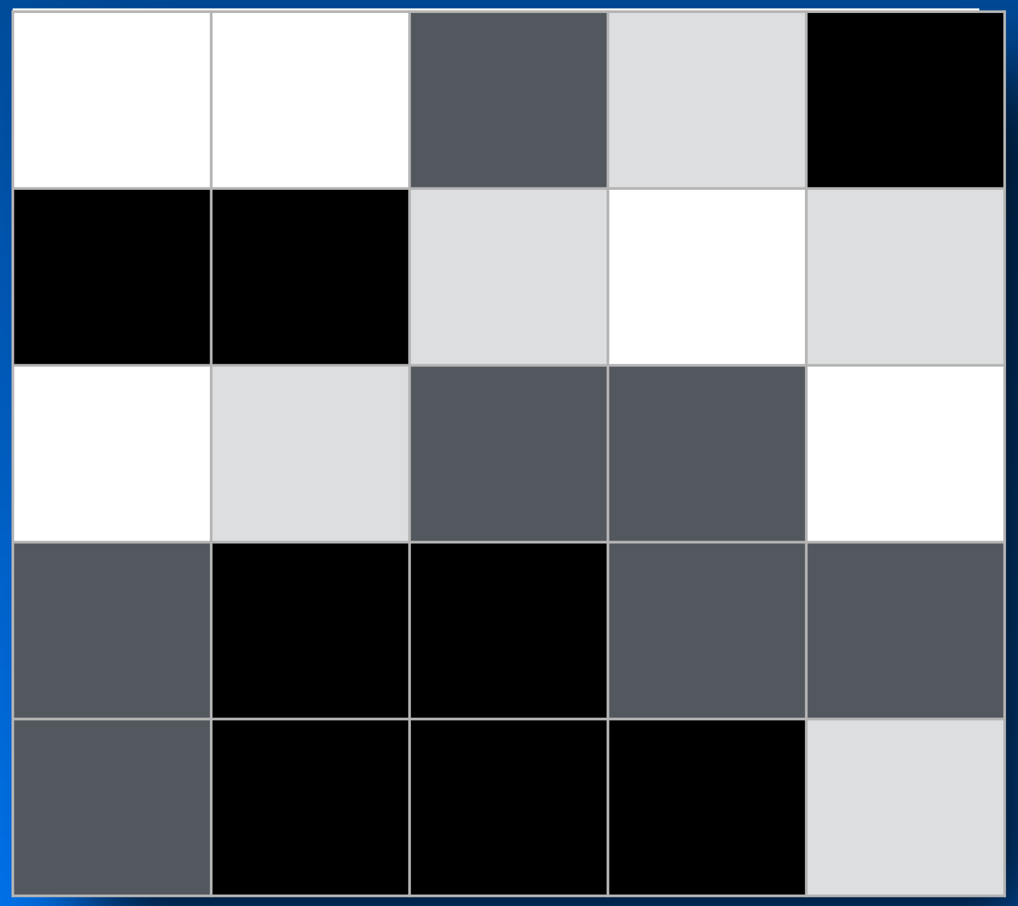
Squeeze Mapping Transformation

$$\begin{pmatrix} 1.50 & 0 & 0 \\ 0 & 0.67 & 0 \\ 0 & 0 & 0.67 \end{pmatrix}$$

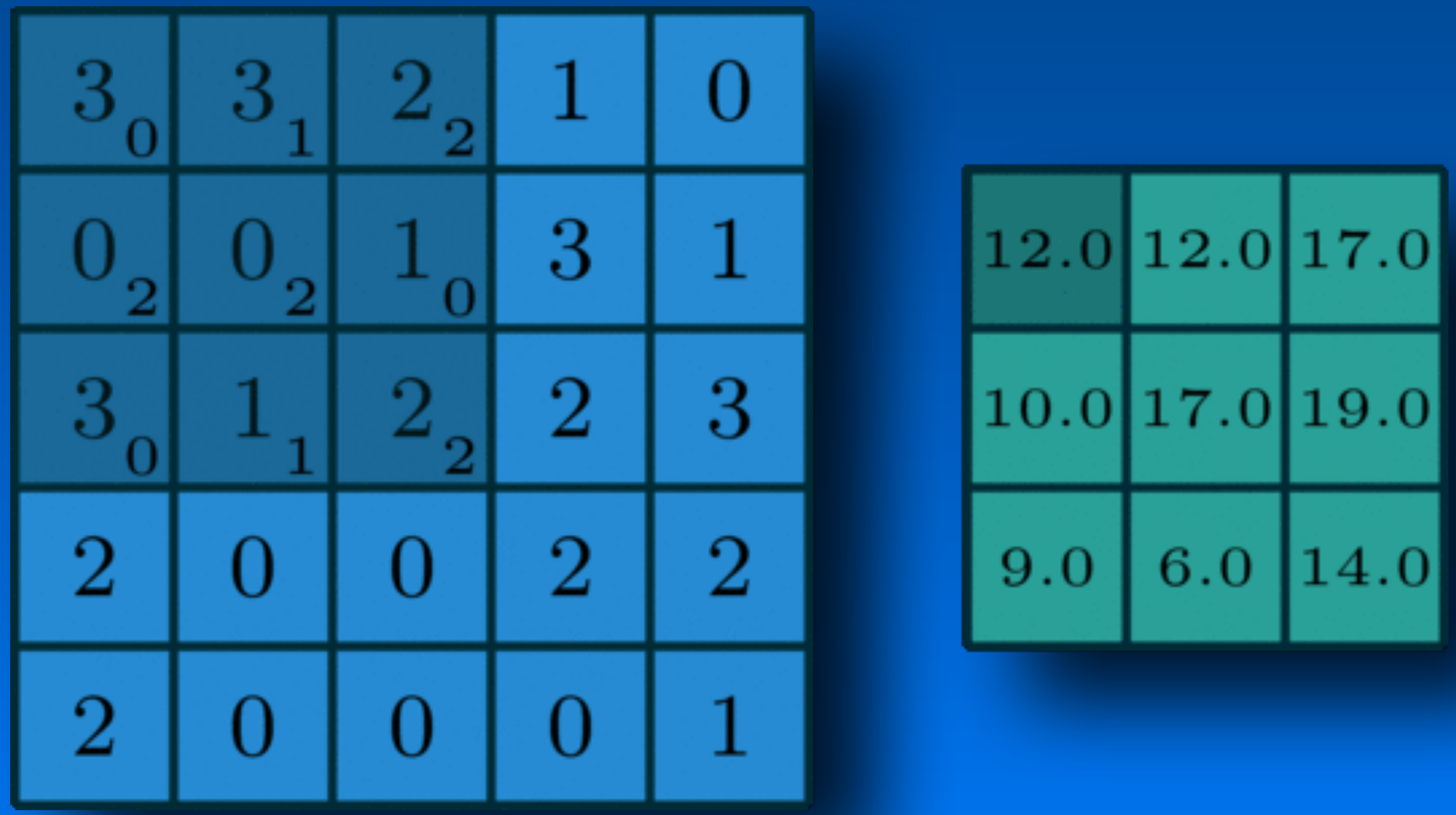


Matrix Transformations

Convolution Filtering - A common signal-processing technique in which a matrix (or “kernel” is “slid” over the signal matrix with the value and geometry of the kernel’s weights being used to boost or suppress aspects of the signal. Convolution filters can also be used to reduce the size of the signal without sacrificing information content, an operation often referred to as “pooling”.



0	1	2
2	2	0
0	1	2

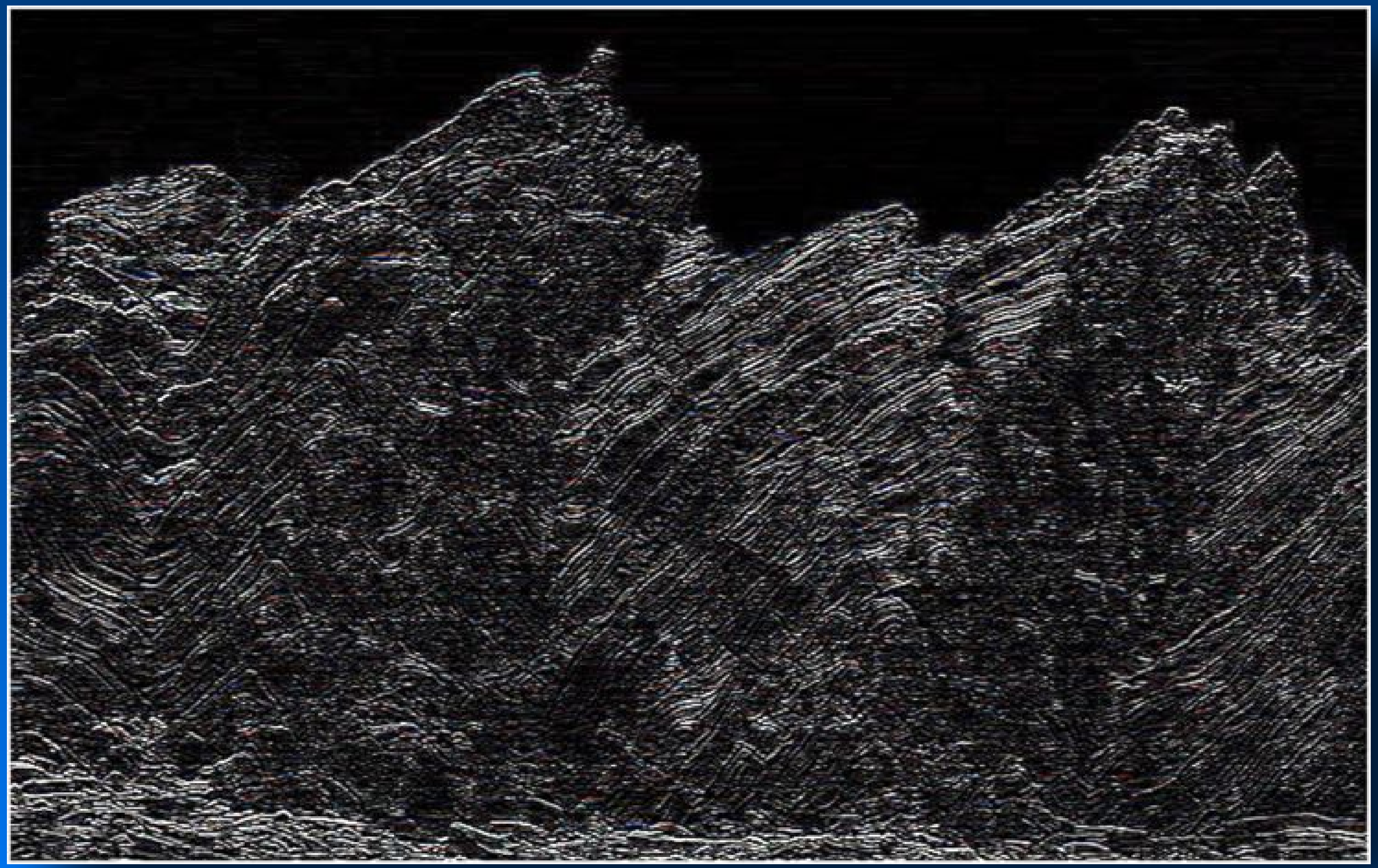


Matrix Transformations

Convolution Filtering

Horizontal Edge Detection

$$\begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$$

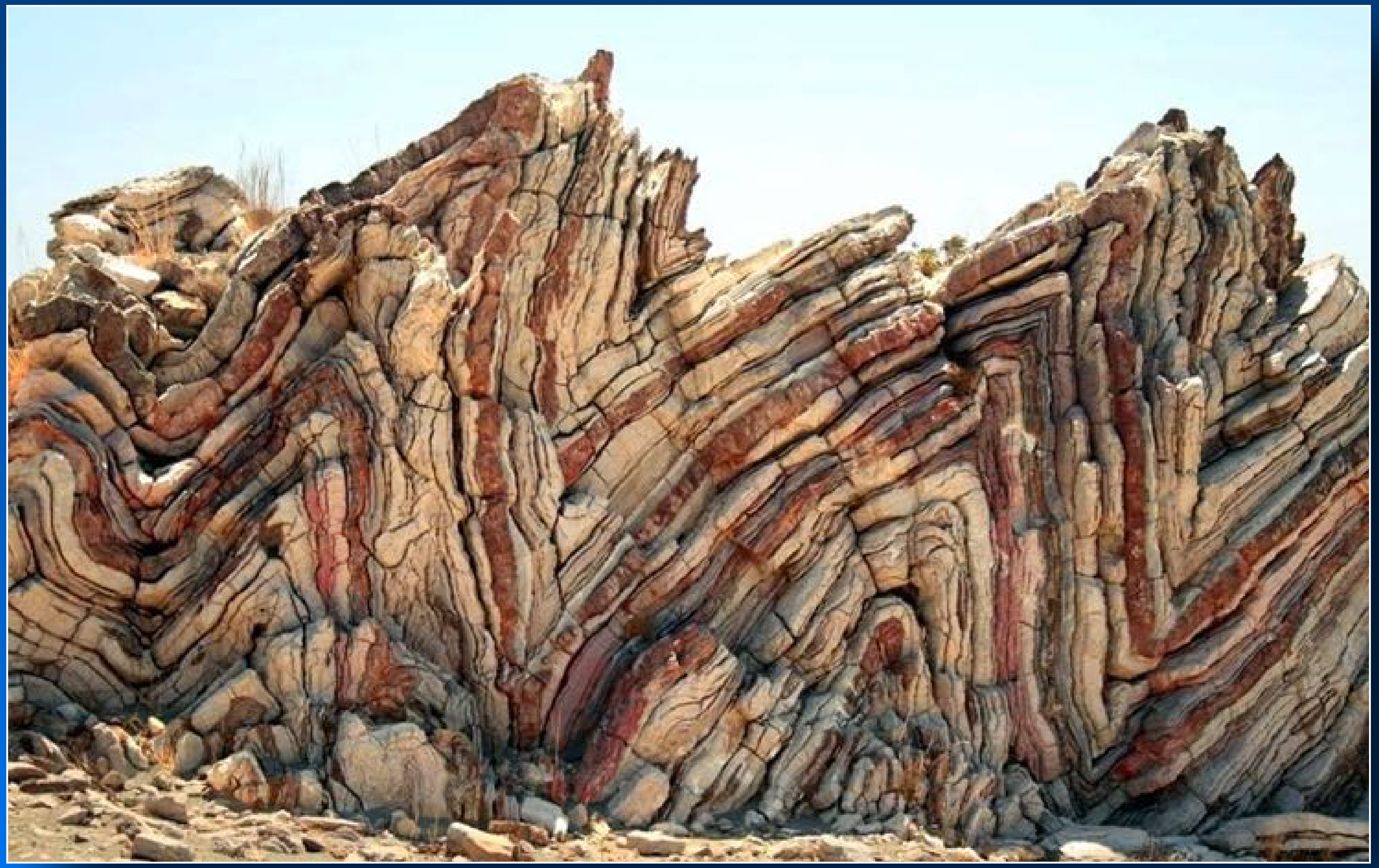


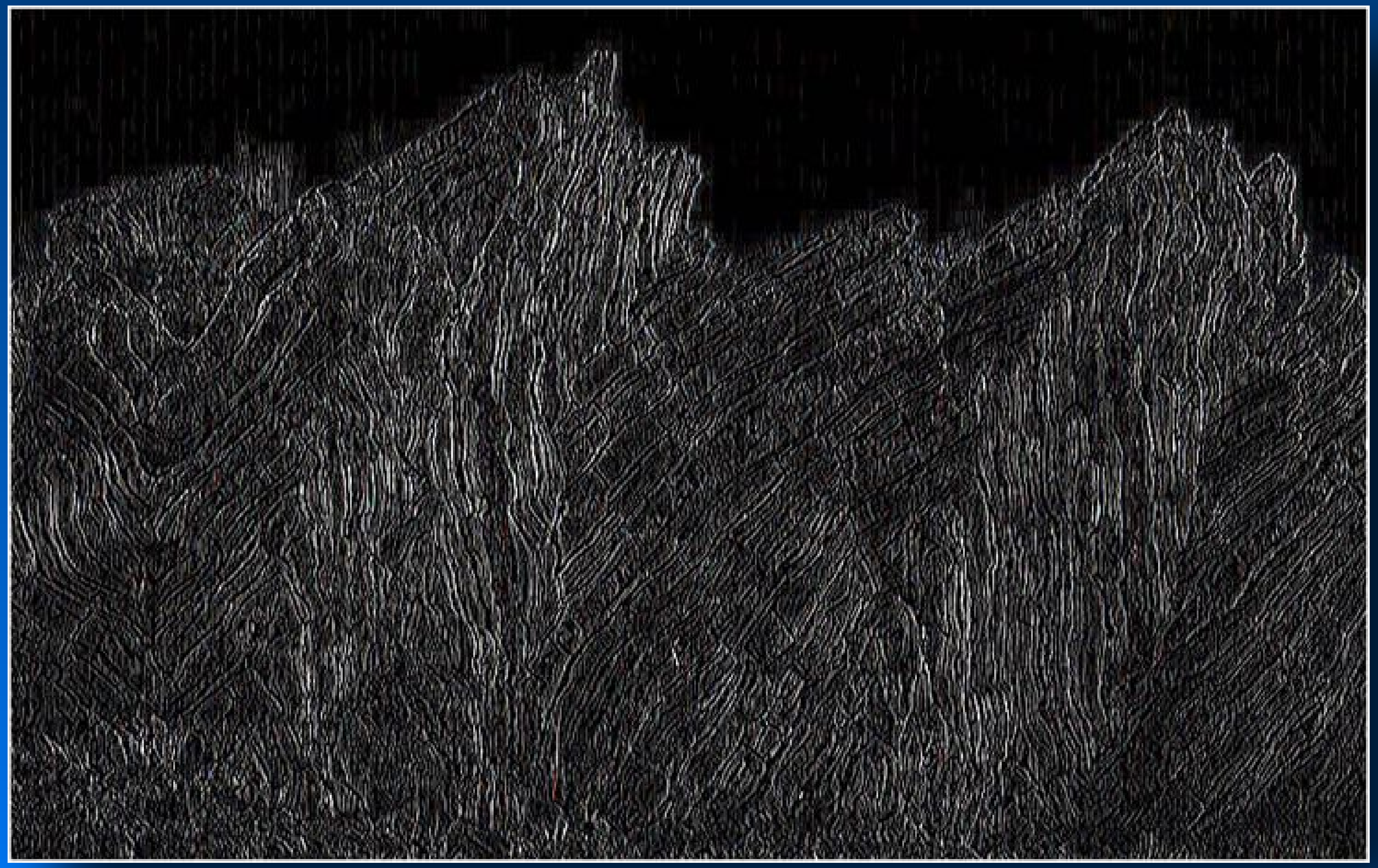
Matrix Transformations

Convolution Filtering

Vertical Edge Detection

$$\begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$





Matrix Transformations

Convolution Filtering

Omnidirectional Edge Detection

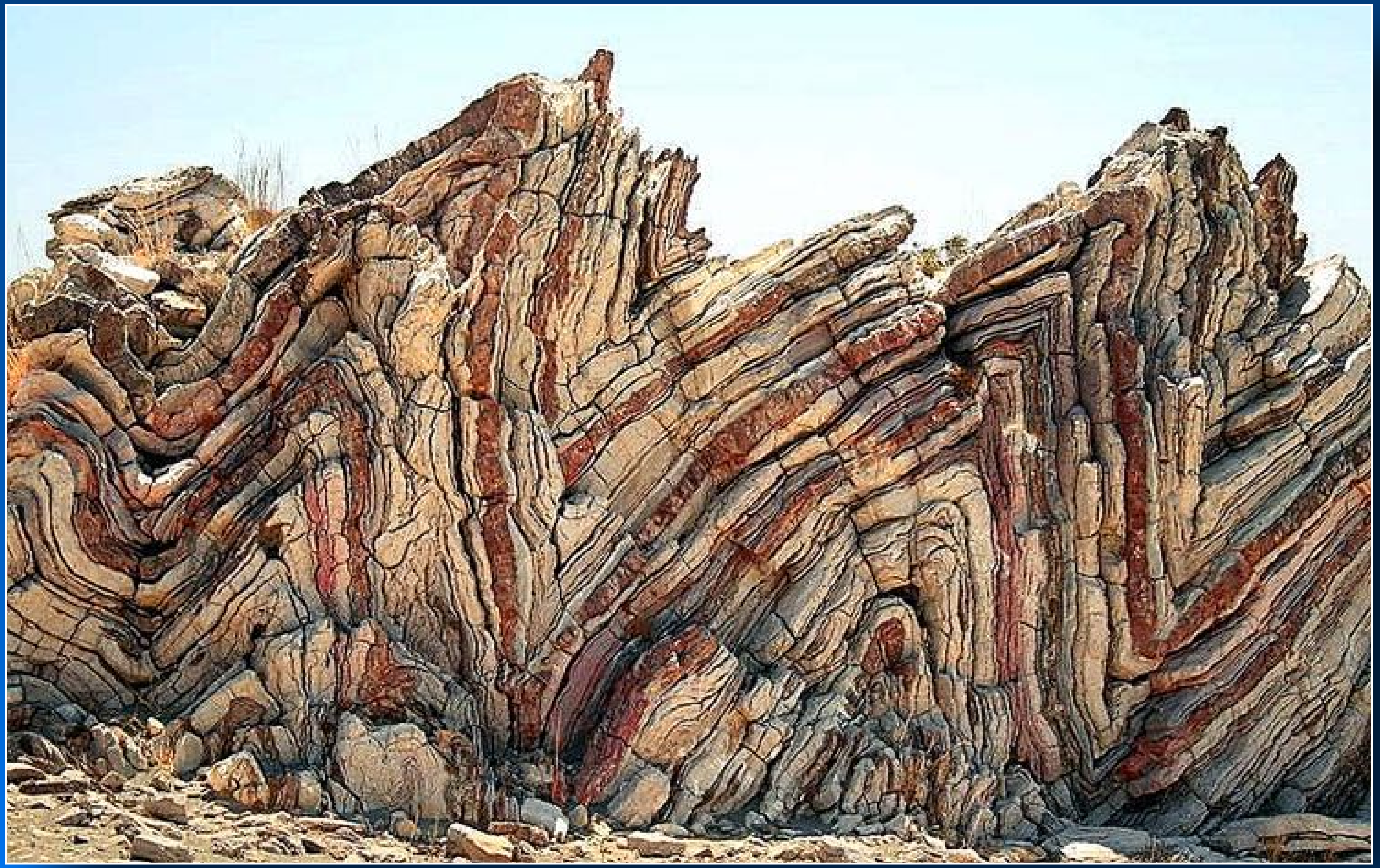
$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

Matrix Transformations

Convolution Filtering

Sharpening Filter

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$



Matrix Manipulations for Earth Scientists

Prof. Norman MacLeod
School of Earth Sciences & Engineering, Nanjing University